The Fundamental Tone $2^{-1/2-it}$ on the Critical Line and the Prime–Zero Duality

— A Note from the Viewpoint of Turing Patterns —

Toshiki Takahashi*
TML independent researcher

August 18, 2026

Abstract

This note is a memorandum organizing, in the form of a paper, a dialogue that began with the question: for which family of waves does a Turing pattern become "ζ-like"? (1) We review the two spectra — the prime side and the zero side — given by Riemann's explicit formula, and their set-to-set Fourier duality (Riemann 1859, von Mangoldt 1895, Weil 1952, together with Dyson's quasicrystal interpretation). (2) We show, by elementary means, that the unique prime wave whose complex amplitude has absolute value exactly $1/\sqrt{2}$ on the critical line is the *fundamental tone* $W(t) = 2^{-1/2-it}$; that under the von Mangoldt normalization this value is unattainable for transcendence-theoretic reasons; and that on the zero side it is unattainable by the lower bound on the height of the zeros. (3) Observing the superposition of the conjugate pair $W, \overline{W}$ — that is, taking the squared absolute value of the complex amplitude of each component — each component carries exactly 1/2; we explain this as the balance point of the exact invariant $|2^{-s}|\,|2^{-(1-s)}| = 1/2$ attached to the mirror pair of the functional equation. (4) We distinguish carefully which of these restatements are theorems and which are mere tautologies. (5) Finally, we sketch the design requirements for a hypothetical Turing system with a ζ-like spectrum, note that to the compiler's knowledge no such construction exists in the literature, and record the historical circumstance that Turing himself treated both subjects — morphogenesis and the computation of the zeros of ζ — in 1952–53 without ever joining them. This note claims no new mathematical results.

1 Introduction: when does a Turing pattern become "ζ-like"?

Turing's diffusion-driven instability [8] is a mechanism by which the dispersion relation $\lambda(k)$, obtained from the linearization of a reaction–diffusion system, becomes positive only near an essentially single critical wavenumber k_c, thereby selecting patterns — stripes and spots — that are crystalline when viewed in wavenumber space (see [16] for a review; Fig. 1, left).

*This note records, in article form, a dialogue held on August 17, 2026 (a user and Claude, Anthropic), compiled into paper form by Claude on the same day; the present English version was prepared from the Japanese original. This is a revised version, correcting errors pointed out in a follow-up dialogue on the same day: the conflation of the ψ and measure ($d\psi$) versions of the explicit formula (Section 1); the omission of the pole contribution in Weil's formula (eq. (4)); the misnomer "self-dual" and the mismatched citation of [24] (Section 1); the opening claim of Section 7.4; and bibliographic details (the title of [7], the in-text reference to [13], the remark on the convergence of the sum in (3)). As stated explicitly in Section 6, this note claims no new mathematical results. All errors are the compiler's. In the present version (August 18, 2026) the author line has been changed, the closing of Section 8 replaced, and eleven figures together with the generating script (Appendix A) added.

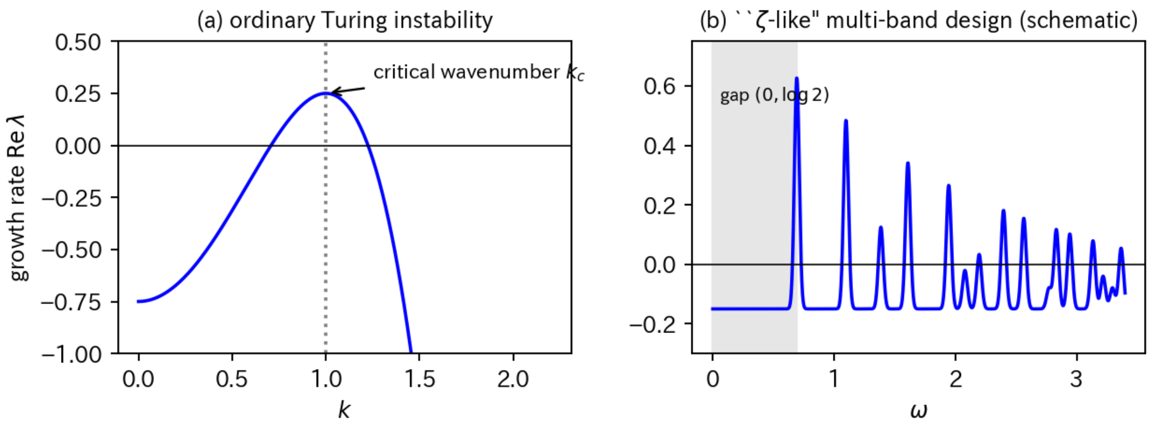

Figure 1: Dispersion relations (schematic). Left: the ordinary Turing instability has positive growth only near a single critical wavenumber k_c, selecting crystalline patterns. Right: the design target for a "ζ-like" system — growth positive only on the multi-band prime comb $\{k \log p\}$ (vertical scale and band widths arbitrary); the gap $(0, \log 2)$ contains no band.

By "ζ-like" we mean, in this note, the contrasting situation: a family of log-periodic waves $\cos(\gamma_n \xi)$ in the logarithmic variable $\xi = \log x$, whose frequency set consists of the imaginary parts of the nontrivial zeros of the Riemann zeta function, $\gamma_1 = 14.134725\ldots$, $\gamma_2 = 21.022040\ldots$, $\gamma_3 = 25.010858\ldots$, with nearly uniform amplitudes — though, as explained below, this uniform-amplitude picture belongs to the measure (differentiated) version of the explicit formula. If all zeros lie on the critical line (that is, assuming the Riemann Hypothesis), the Riemann–von Mangoldt explicit formula (Section 2) takes the wave-packet form (Fig. 2)

$$\psi(x) - x \;\approx\; -2\sqrt{x} \sum_{n \geq 1} \frac{\cos(\gamma_n \log x - \theta_n)}{|\rho_n|}, \qquad \theta_n = \arg \rho_n. \tag{1}$$

Here the weight of each zero wave decays as $1/|\rho_n| \approx 1/\gamma_n$, so the amplitudes in (1) are not uniform, and what this sum reconstructs is the staircase of $\psi(x)$ — the jumps of height $\Lambda(p^k) = \log p$ at the prime powers $x = p^k$. Sharp (delta-like) spikes at the prime powers arise instead in the measure version obtained by differentiating (1) with respect to ξ — the $d\psi$ side, where each zero contributes with equal weight through the uniform-amplitude sum $\sum_n \cos(\gamma_n \xi)$ (Fig. 3). Formally (under the Riemann Hypothesis, in the sense of distributions),

$$\sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}}\, \delta(\xi - \log n) \;\approx\; \mathrm{e}^{\xi/2} - 2 \sum_{n \geq 1} \cos(\gamma_n \xi) + (\text{smooth decaying terms}),$$

and, read dually, the spectrum of the family of "prime waves" of frequency $k \log p$ and amplitude $(\log p)\, p^{-k/2}$ — the weighted prime comb on the left-hand side — is concentrated at the ordinates $\{\gamma_n\}$ of the zeros.

This frequency set is neither equally spaced (a crystal) nor rationally related (a superposition of commensurate periods). Its spacing statistics agree with those of the GUE random-matrix ensemble (Montgomery–Odlyzko [12, 15]; Fig. 4). Dyson proposed to view this point set as an example of a one-dimensional quasicrystal — a pure point measure whose Fourier transform is again supported on a point set [22]. The duality, however, is not self-duality but a mutual duality

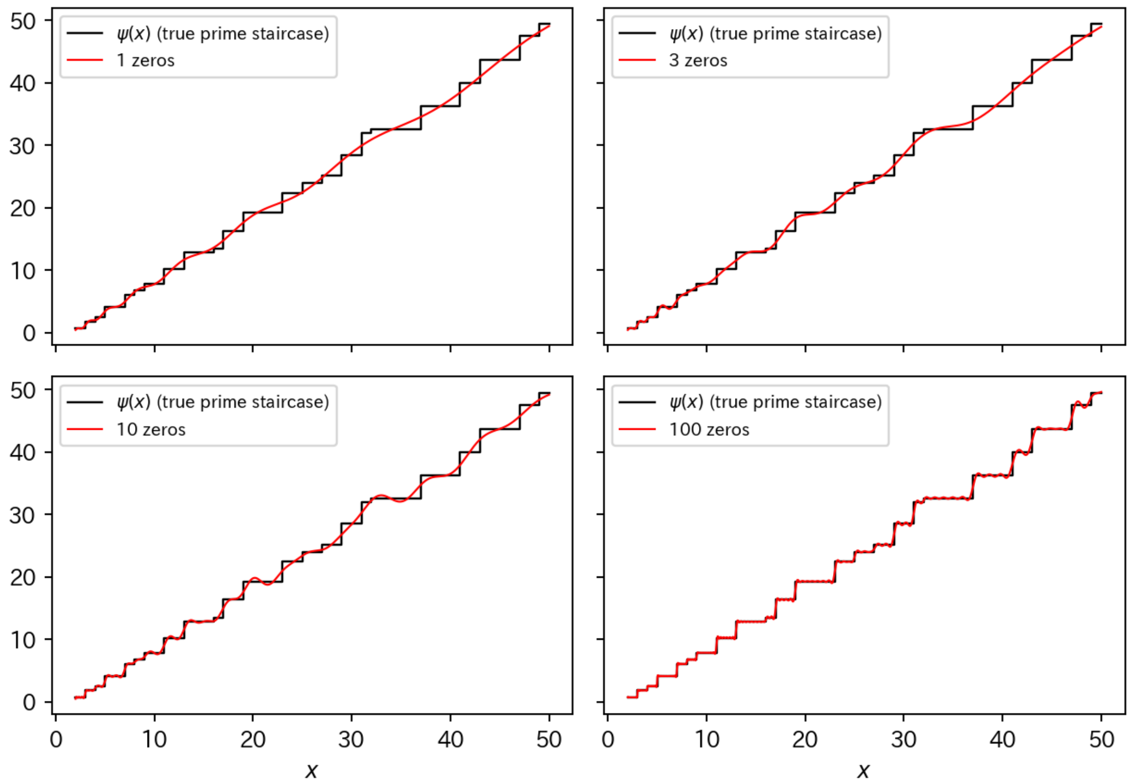

Figure 2: Partial sums of the explicit formula (1): the interference of wave packets built from $1, 3, 10, 100$ zeros reconstructs the prime staircase $\psi(x)$.

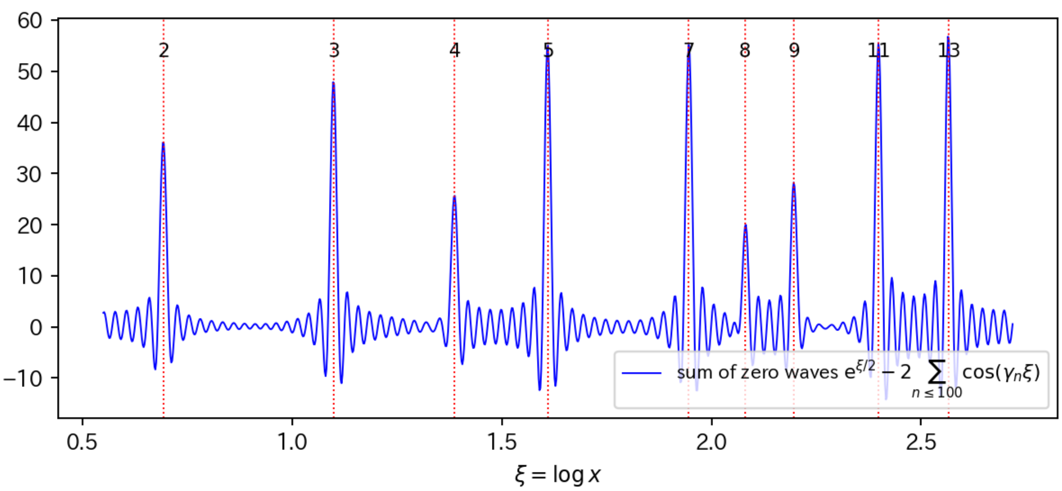

Figure 3: The measure ($d\psi$) side. The uniform-amplitude sum of zero waves $e^{\xi/2} - 2\sum_{n\le 100}\cos(\gamma_n\xi)$ raises sharp spikes at the prime-power positions $\xi = \log n$ (red dotted lines, labelled by n) — the direction of the duality in which the zeros "hear" the primes.

of two sets: the Fourier transform of the set of zeros is supported not on that set itself but on the different set $\{\pm k \log p\}$ (and Dyson's definition, too, requires only that the Fourier transform of a point measure be again a point measure, not self-duality). As a related numerical study, Torquato et al. showed that the structure factor of the primes themselves, placed on the integer line rather than on the logarithmic comb $\{k \log p\}$, exhibits dense Bragg peaks; but the order found there is "effectively limit-periodic", and the authors themselves distinguish it from quasicrystallinity [24].

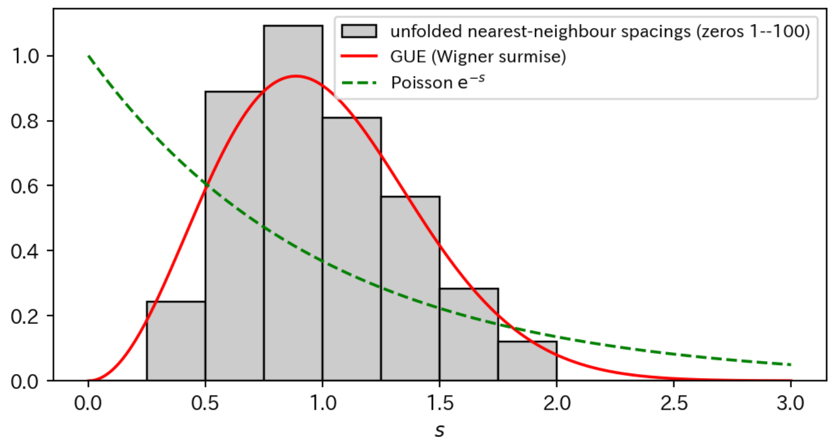

Figure 4: Histogram of the unfolded nearest-neighbour spacings of the first 100 zeros against the GUE Wigner surmise $p(s) = \frac{32}{\pi^2} s^2 \mathrm{e}^{-4s^2/\pi}$ and the Poisson law. The sample is small (99 gaps), but the strong level repulsion and the GUE shape are visible.

We emphasize that there is no established theory by which the ordinary Turing mechanism itself produces such a spectrum. To make it ζ-like one needs both (a) a nonlocal coupling kernel designed so that the growth rate is positive only on the comb-shaped multi-band $\{k \log p\}$, and (b) a scale-invariant substrate (such as an exponentially growing domain) carrying the log-periodicity (Section 7; Fig. 1, right). The purpose of this note is to organize the observations obtained in a dialogue and to make clear the status of each claim: classical theorem, elementary proposition, exact identity, tautology, or hypothesis. Every figure in this note can be reproduced with the single script of Appendix A.

2 The explicit formula and the two spectra

The starting point is the Euler product [1]

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}, \qquad \operatorname{Re} s > 1,$$

whose logarithm expands into a sum of waves of frequency $k \log p$ and amplitude $\frac{1}{k} p^{-k\sigma}$:

$$\log \zeta(s) = \sum_p \sum_{k \geq 1} \frac{1}{k}\, p^{-ks} = \sum_{p,k} \frac{1}{k}\, p^{-k\sigma}\, \mathrm{e}^{-ikt \log p}, \qquad s = \sigma + it, \tag{2}$$

absolutely convergent for $\operatorname{Re} s > 1$. By unique factorization, $p^k = q^\ell \Rightarrow (p, k) = (q, \ell)$, so the frequencies are pairwise distinct, and by the mean value of Bohr's theory of almost periodic functions [6],

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \log\zeta(\sigma+it)\, \mathrm{e}^{i\omega t}\, dt,$$

each coefficient is recovered uniquely for $\sigma > 1$. On the critical line the series (2) itself does not converge, but each term is defined termwise as the restriction of the entire function $\frac{1}{k}p^{-ks}$ (Section 3).

The explicit formula stated by Riemann [2] and proved by von Mangoldt [5] gives, for $x > 1$ not a prime power,

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log 2\pi - \tfrac{1}{2}\log\bigl(1 - x^{-2}\bigr), \tag{3}$$

where ψ is the Chebyshev function and the sum runs over the nontrivial zeros; the sum over zeros is conditionally convergent and is to be taken in the symmetric sense $\lim_{T\to\infty}\sum_{|\operatorname{Im}\rho|<T}$ (for the classical material of this section see [13]). Grouping the zeros in conjugate pairs $\rho, \bar\rho$ and assuming $\rho = \beta + i\gamma$ lies on the critical line ($\beta = 1/2$),

$$\frac{x^{\rho}}{\rho} + \frac{x^{\bar\rho}}{\bar\rho} = \frac{2\sqrt{x}}{|\rho|}\cos\bigl(\gamma\log x - \arg\rho\bigr),$$

which yields (1) (in general the factor is x^{β}).

Weil [9] recast this symmetrically as an identity of distributions against test functions: for suitable even h with Fourier transform g, and up to normalization,

$$\sum_n h(\gamma_n) = h\Bigl(\frac{i}{2}\Bigr) + h\Bigl(-\frac{i}{2}\Bigr) + \text{archimedean terms} - \sum_{p,k} (\log p)\, p^{-k/2}\Bigl[g(k\log p) + g(-k\log p)\Bigr]. \tag{4}$$

The terms $h(\pm i/2)$ at the head of the right-hand side are the contribution of the pole of ζ at $s = 1$ (together with its mirror $s = 0$ under the functional equation); the main term x of the explicit formula (3) originates here. What is essential is that the correspondence is a *set-to-set* Fourier duality $\{\gamma_n\} \leftrightarrow \{\pm k\log p\}$: there is no term-by-term correspondence between an individual prime wave and an individual zero. No prime wave, and no zero, has a designated partner; both directions of the duality are visualized in Figs. 3 and 5.

The prospect of viewing the zeros as the eigenfrequencies of some system originates in the Hilbert–Pólya conjecture (transmitted orally; there is no publication), was sharpened into the quantum-chaos reading — prime waves as periodic orbits, the explicit formula as a semiclassical trace formula — by Berry [14] and Berry–Keating [18], and a related trace formula was given by Connes [19].

3 The fundamental tone $W(t)$ and its exact uniqueness

Definition 3.1. *On the critical line* $\operatorname{Re}s = 1/2$ *define the prime wave*

$$W_{p,k}(t) := \frac{1}{k}\, p^{-k(1/2+it)} = \underbrace{\frac{1}{k}\, p^{-k/2}}_{\textit{complex amplitude } a_{p,k}} \mathrm{e}^{-it(k\log p)},$$

the restriction to the critical line of the entire function $\frac{1}{k}p^{-ks}$*; each term is thus rigorously defined.*

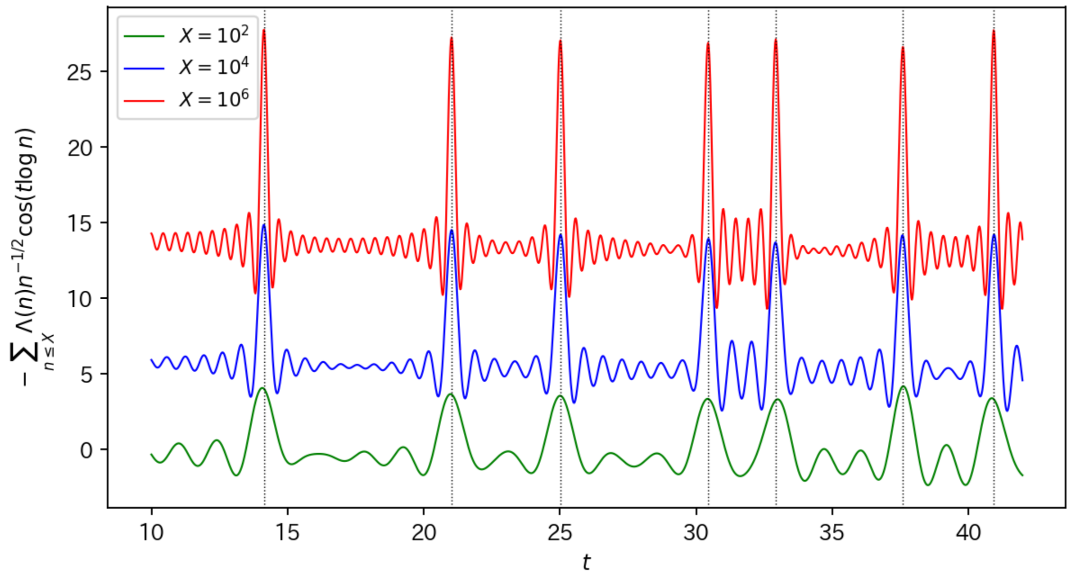

Figure 5: The dual direction — the primes "hear" the zeros. The weighted prime sum $\sum_{n\le X} \Lambda(n)n^{-1/2}\cos(t\log n)$ with the pole main term $\mathrm{Re}[X^{1/2-it}/(\frac{1}{2}-it)]$ removed and the sign reversed (curves offset vertically for clarity). As $X = 10^2 \to 10^6$ the peaks sharpen exactly at the zeros γ_n (dotted lines): no individual prime wave points at any zero, but the ensemble points at all of them.

Proposition 3.2 (Uniqueness of the amplitude $1/\sqrt{2}$)**.** *$a_{p,k} = 2^{-1/2}$ holds if and only if $(p,k) = (2,1)$.*

Proof. For $k = 1$, the map $p \mapsto p^{-1/2}$ is strictly decreasing on the primes, so $p = 2$ is the only solution. For $k \ge 2$, $\frac{1}{k}p^{-k/2} \le \frac{1}{2}\cdot 2^{-1} = \frac{1}{4} < 2^{-1/2}$, so there is no solution. □

The fundamental tone so determined,

$$W(t) = 2^{-(1/2+it)} = \frac{\sqrt{2}}{2}\,\mathrm{e}^{-it\log 2}, \tag{5}$$

satisfies the exact identity $|W(t)| = 2^{-1/2}$ for all real t. Its frequency $\log 2 = 0.693147\ldots$ is the minimum of the entire prime spectrum (the lowest tone), its period is $2\pi/\log 2 = 9.06472\ldots$, and its amplitude is the positive root of $2x^2 = 1$, an algebraic number (Fig. 6, top).

Proposition 3.3 (Impossibility under the von Mangoldt normalization)**.** *There exist no prime p and integer $k \ge 1$ with $(\log p)\,p^{-k/2} = 2^{-1/2}$.*

Proof. Squaring and rearranging gives $2(\log p)^2 = p^k$. By the Hermite–Lindemann theorem [4], e^{α} is transcendental for every nonzero algebraic α; contrapositively, $\log p$ is transcendental for every integer $p \ge 2$. If $(\log p)^2$ were algebraic then, the algebraic numbers being closed under the extraction of roots, $\log p$ would be algebraic — a contradiction. Hence the left-hand side $2(\log p)^2$ is transcendental, while the right-hand side p^k is an integer. □

Proposition 3.4 (Impossibility on the zero side)**.** *No nontrivial zero $\rho = \beta + i\gamma$ (with $0 < \beta < 1$, $\gamma > 0$) satisfies $1/|\rho| = 2^{-1/2}$.*

Proof. One would need $|\rho| = \sqrt{2}$, but $|\rho| \geq \gamma \geq \gamma_1 = 14.134725\ldots > \sqrt{2}$. That no zeros exist up to height 14 is established by classical numerical verification (by now the verification extends to height $3 \cdot 10^{12}$ [25]) and does not require the Riemann Hypothesis. □

Remark 3.5 (A numerical coincidence). For the first zero, $1/|\rho_1| = 0.0707035\ldots \approx \frac{1}{10} \cdot 2^{-1/2} = 0.0707107\ldots$ This is a numerical accident caused by $\gamma_1 = 14.1347\ldots$ lying close to $10\sqrt{2} = 14.1421\ldots$; no meaning for it is known.

Remark 3.6 (On normalizations). The uniqueness above refers to the normalization $a_{p,k} = \frac{1}{k}p^{-k/2}$ (the $\Pi(x)$-type convention of Riemann, in which the prime power p^k is counted with weight $1/k$). In this normalization the fundamental tone 2 has the largest amplitude, $2^{-1/2} = 0.7071\ldots$ Under the von Mangoldt normalization $\Lambda(n)/\sqrt{n} = (\log p)\, p^{-k/2}$, however, the function $x \mapsto (\log x)/\sqrt{x}$ attains its maximum at $x = \mathrm{e}^2 = 7.389\ldots$, so the strongest wave is $n = 7$ (amplitude $0.735485\ldots$), not the fundamental; see Table 1. In either normalization, by Propositions 3.2 and 3.3, the only wave whose amplitude is exactly $1/\sqrt{2}$ is $(2, 1)$ in the $\frac{1}{k}p^{-k/2}$ normalization (Figs. 6 and 7).

Table 1: The first few prime waves (on the critical line).

$n = p^k$	frequency $k \log p$	amplitude $\frac{1}{k}p^{-k/2}$	amplitude $\Lambda(n)/\sqrt{n}$
2	0.69315	$0.70711 = 1/\sqrt{2}$	0.49013
3	1.09861	0.57735	0.63428
$4 = 2^2$	1.38629	0.25000	0.34657
5	1.60944	0.44721	0.71976
7	1.94591	0.37796	0.73549
$8 = 2^3$	2.07944	0.11785	0.24506
$9 = 3^2$	2.19722	0.16667	0.36620

4 The spectral gap and Euler's constant

Since the frequencies of the prime spectrum satisfy $k \log p \geq \log 2$, the spectrum has a gap $(0, \log 2)$. The Euler(–Mascheroni) constant $\gamma_E = 0.577216\ldots$ is smaller than $\log 2 = 0.693147\ldots$ and falls into this gap below the fundamental (Fig. 8). Directly: $k \log p = \gamma_E$ would require $p^k = \mathrm{e}^{\gamma_E} = 1.781072\ldots \in (1, 2)$, whereas a prime power is an integer ≥ 2, so there is no solution.[1] Had a prime wave of that frequency existed, its amplitude would have been $\mathrm{e}^{-\gamma_E/2} = 0.7493\ldots > 2^{-1/2}$ — the strongest wave, exceeding the fundamental; the world of primes leaves that seat empty.

The actual residence of γ_E is on the side of the continuous background, not of the discrete spectrum. (i) It is the constant term of the Laurent expansion of ζ at $s = 1$: $\zeta(s) = \frac{1}{s-1} + \gamma_E + O(s-1)$. (ii) It appears in Mertens' theorem [3], $\prod_{p \leq x}(1 - 1/p)^{-1} \sim \mathrm{e}^{\gamma_E} \log x$, as the coefficient bundling the collective effect of all prime waves. (iii) It sits in the archimedean term (the Γ-factor) of Weil's formula (4). In particular the digamma function $\psi_\Gamma = \Gamma'/\Gamma$ (the notation is chosen to avoid confusion with the Chebyshev function ψ) satisfies, at the center $s = 1/2$ of the critical line, the exact identity

$$\psi_\Gamma\left(\tfrac{1}{2}\right) = -\gamma_E - 2\log 2. \tag{6}$$

[1] If "Euler's constant" is read as Napier's constant e, the same holds: $p^k = \mathrm{e}^{\mathrm{e}} = 15.154\ldots$ is not an integer.

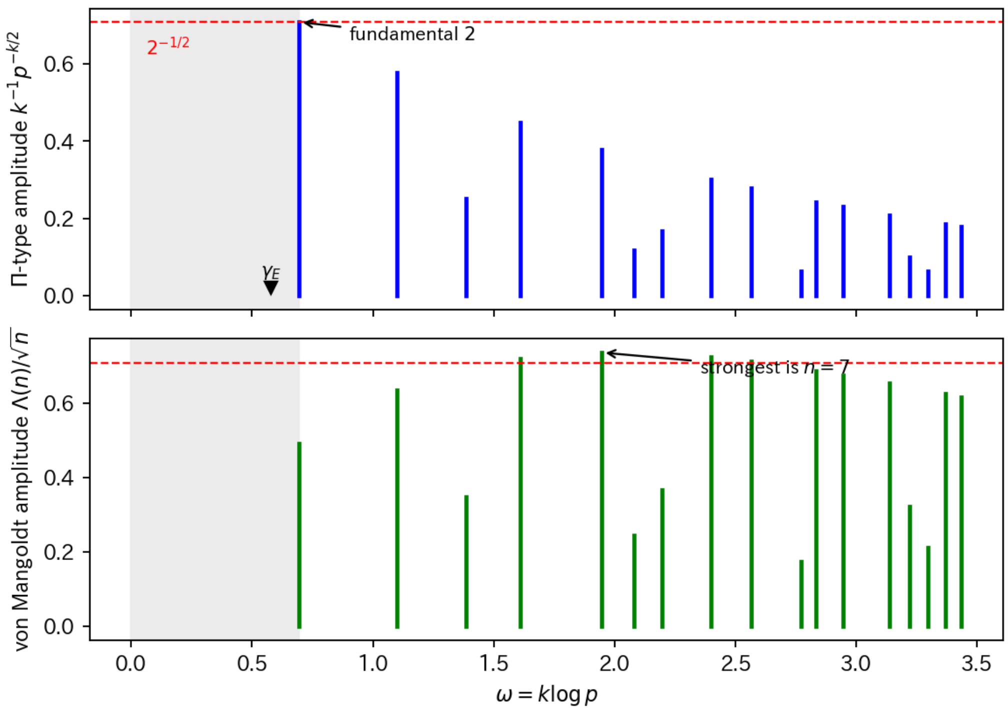

Figure 6: The two normalizations of the prime comb (Table 1 as a picture, $\omega \leq 3.45$). Top: in the Π-type weights $k^{-1}p^{-k/2}$ only the fundamental 2 reaches the level $2^{-1/2}$ (red dashed; Proposition 3.2). Bottom: under the von Mangoldt weights the strongest line is $n = 7$ (Remark 3.6). The grey band is the gap $(0, \log 2)$; the marker ▼ is γ_E.

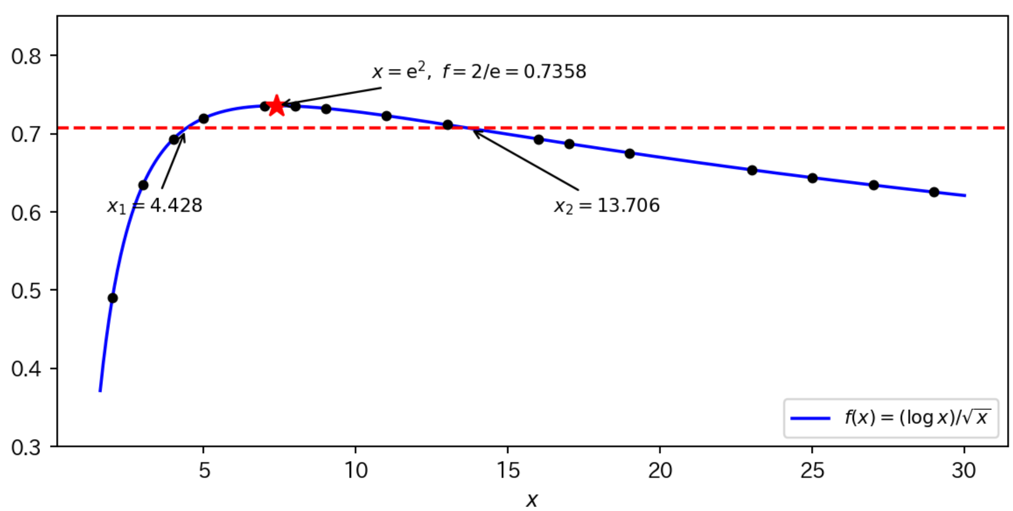

Figure 7: The transcendence picture for the von Mangoldt normalization (Proposition 3.3). The curve $f(x) = (\log x)/\sqrt{x}$ crosses the level $2^{-1/2}$ (red dashed) at two points x_1, x_2 (values in the figure), neither a prime power; no prime power (black dots) lies exactly on the line. The maximum is at $x = e^2$ ($\star$).

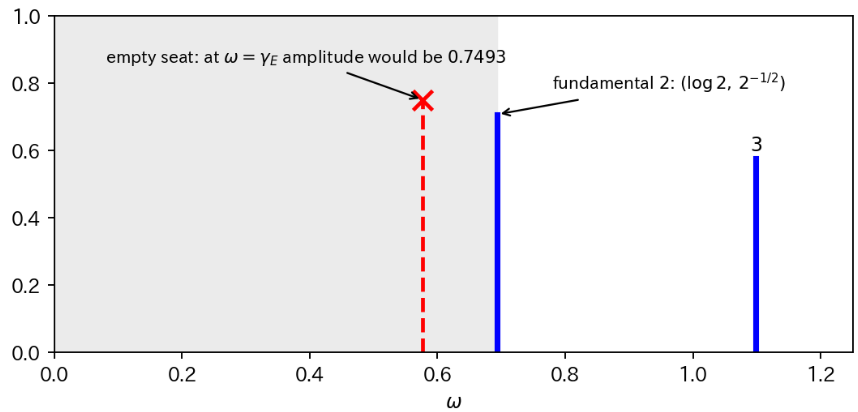

Figure 8: The empty seat below the fundamental. No prime wave exists at $\gamma_E = 0.5772$ inside the gap $(0, \log 2)$ (grey), since $e^{\gamma_E} \in (1,2)$; had one existed, its amplitude $e^{-\gamma_E/2} = 0.7493$ (red dashed, $\times$) would have exceeded the fundamental $2^{-1/2}$.

It is noteworthy that the background constant γ_E and (twice) the fundamental frequency $\log 2$ are joined by a single classical identity at the very center of the critical line (Fig. 9).

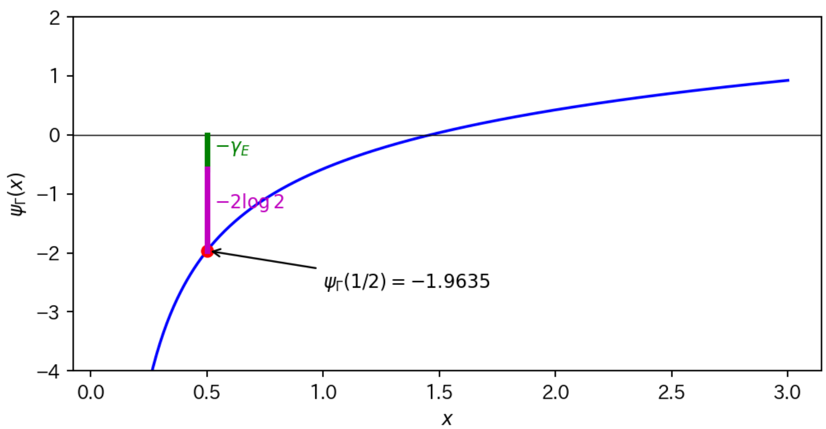

Figure 9: The identity (6): the value of the digamma function ψ_Γ at the center $x = 1/2$ of the critical line decomposes exactly into the background constant $-\gamma_E$ (green) and twice the fundamental frequency, $-2\log 2$ (magenta).

5 Observation and 1/2: the mirror-pair identity

To obtain a real pattern field one superposes the fundamental W with its complex conjugate (the negative-frequency component) $\overline{W}$:

$$\varphi(t) = W(t) + \overline{W(t)} = \sqrt{2}\cos(t\log 2) = 2^{-1/2}e^{-i\omega t} + 2^{-1/2}e^{+i\omega t}, \qquad \omega = \log 2. \tag{7}$$

Observation 5.1 (The observed value 1/2)**.** *The complex amplitudes of the two components of the conjugate pair are $a_\pm = 2^{-1/2}$, and their squared absolute values — the "observation" in the* *analogy with the Born rule — are*

$$|a_+|^2 = |a_-|^2 = \frac{1}{2}.$$

Since the frequencies $\pm\omega$ are distinct, Parseval's identity gives the mean power $\langle \varphi^2 \rangle = |a_+|^2 + |a_-|^2 = 1$: the unit total power is shared exactly half-and-half by the two components. This is isomorphic to the measurement probabilities $(1/2, 1/2)$ of the normalized equal-weight superposition $(|+\rangle + |-\rangle)/\sqrt{2}$.

This 1/2 has an exact parent, valid for every s, coming from the symmetry of the functional equation.

Proposition 5.2 (Invariant of the mirror pair)**.** *For the mirror pair $\left(2^{-s}, 2^{-(1-s)}\right)$ determined by the symmetry $s \leftrightarrow 1-s$ of the functional equation, one has, for all $s = \sigma + it \in \mathbb{C}$,*

$$\left|2^{-s}\right| \cdot \left|2^{-(1-s)}\right| = 2^{-\sigma}\, 2^{\sigma-1} = \frac{1}{2}. \tag{8}$$

The two factors are equal if and only if $\sigma = 1/2$, in which case their common value is $2^{-1/2}$, whose square coincides with the invariant $1/2$.

Proof. $|2^{-s}| = 2^{-\sigma}$, so the product equals 2^{-1} independently of σ; the equality condition is $2^{-\sigma} = 2^{\sigma-1} \iff \sigma = 1/2$. □

Thus 1/2 is the invariant of the mirror pair — the same on every vertical line — and the critical line is the unique vertical line on which the two factors balance. At the balance point the arithmetic–geometric mean inequality becomes an equality, and the "observed value" of each factor is the invariant itself: exactly 1/2 (Fig. 10).

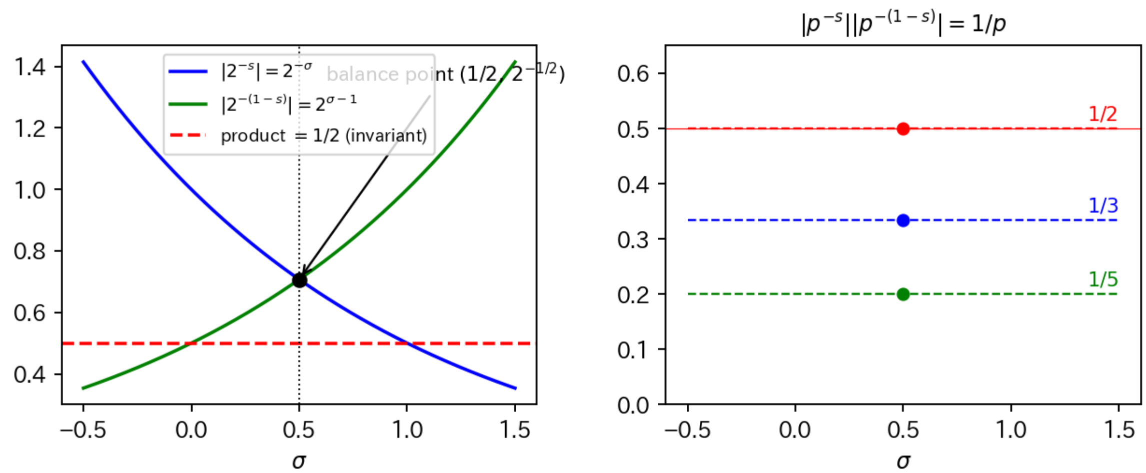

Figure 10: The mirror-pair invariant (Proposition 5.2). Left: $|2^{-s}|$ and $|2^{-(1-s)}|$ balance only at $\sigma = 1/2$ (common value $2^{-1/2}$), while their product is constantly 1/2. Right: for a general prime the product is $1/p$; the invariant coincides numerically with the critical abscissa 1/2 only for $p = 2$ (Remark 5.3).

Remark 5.3. For a general prime p one has $|p^{-s}|\,|p^{-(1-s)}| = 1/p$; the balance point is again $\sigma = 1/2$, and the squared factor there is $1/p$. The invariant coincides numerically with the critical abscissa $1/2$ only when $p^{-1} = 1/2$, that is, for $p = 2$. In this sense the fundamental tone is the unique prime wave whose observed value reads off the coordinate of the critical line.

Remark 5.4. The language of the Born rule is used here as an analogy: no particular Hilbert space or self-adjoint operator is being constructed. The motivation is the Hilbert–Pólya prospect of Section 2.

6 What is a theorem here, and what is not

This section is the crux of the note. Since the maps $\sigma \mapsto 2^{-\sigma}$, $\sigma \mapsto 2^{-2\sigma}$, $\sigma \mapsto 2^{1-2\sigma}$ are all injective, the statements

"amplitude $= 1/\sqrt{2}$", "observed value $|a|^2 = 1/2$", "total power $= 1$", "balance of the mirror pair"

are all reparametrizations of "$\sigma = 1/2$" and contain no information beyond a logically equivalent rewording. In particular the restatement

Riemann Hypothesis $\iff$ every nontrivial zero lies on the unique line on which the observed value of the fundamental tone is exactly $1/2$

is aesthetically pleasing, but adds no arithmetic content whatsoever to the standard statement. A rewording of the same kind can be manufactured from any injective function f, as "the line where $f(\sigma) = f(1/2)$". Consequently, reformulations of this kind have no "first discoverer" — they are not discoveries but rewritings. By contrast, equivalent criteria with genuine content are known: Nyman–Beurling [7, 11], Li [17], and Lagarias [21].

What does hold rigorously in this note is the following: the classical framework of Section 2 (the Euler product, the explicit formula (3), the duality (4)); Propositions 3.2–3.4 (elementary uniqueness and impossibility); the identity (8) of Proposition 5.2; and the identity (6). The true reason the critical line is special is none of these rewordings, but the fact that it is the axis of symmetry of the completed function, $\xi(s) = \xi(1-s)$ (Riemann [2]). Why the zeros stand upon that axis remains unresolved.

7 A hypothetical ζ-like Turing system: design requirements and current status

7.1 What happens for subfamilies

That the fundamental has no designated partner becomes evident from the behaviour of partial superpositions (Fig. 11).

(i) W with its conjugate alone: $\varphi(t) = \sqrt{2}\cos(t\log 2)$, a crystal of a single period.

(ii) The tower of 2 (the fundamental with all its overtones):

$$\sum_{k\geq 1} \frac{1}{k}\, 2^{-k(1/2+it)} = -\log\bigl(1 - 2^{-1/2-it}\bigr).$$

Since $|2^{-1/2-it}| = 2^{-1/2} < 1$, this Euler factor has no zeros and is exactly periodic with period $2\pi/\log 2$. A single prime — a single periodic orbit — is still only a crystal.

(iii) Two primes $\{2, 3\}$: by unique factorization, $\log 2$ and $\log 3$ are linearly independent over $\mathbb{Q}$; the superposition exhibits aperiodic (quasiperiodic) beats, but no zeros arise yet.

(iv) All prime powers with the ζ-weights: the series (2) does not converge on the critical line, but the real part of its analytic continuation, $\log |\zeta(1/2 + it)|$, diverges logarithmically to $-\infty$ at $t = \gamma_n$. Only here — as singularities of the full ensemble — do the zeros come into focus.

Hence the "partner" of W is not any individual zero wave but the totality $\{e^{i\gamma_n \tau}\}$, and the correspondence is only the set-to-set Fourier duality of Section 2.

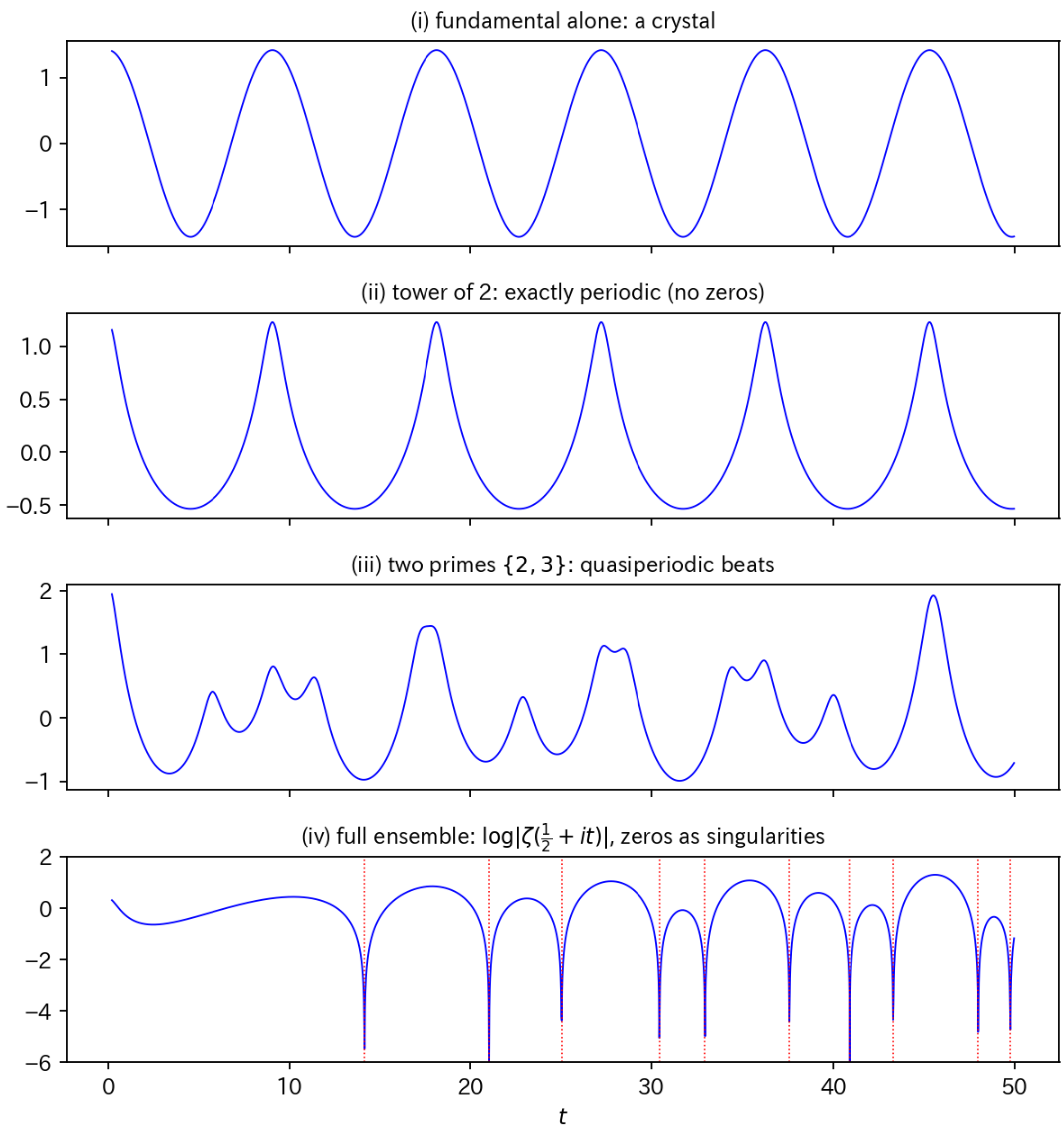

Figure 11: The subfamilies of Section 7.1 (i)–(iv): the fundamental alone (a crystal) $\to$ the tower of 2 (exactly periodic, no zeros) $\to$ two primes (quasiperiodic beats, still no zeros) $\to$ the full ensemble $\log |\zeta(\frac{1}{2} + it)|$, where the zeros (red dotted) first come into focus as singularities.

7.2 A sketch of the design requirements

In the vocabulary of the Turing mechanism the requirements read as follows. Introduce a nonlocal coupling kernel into a two-component activator–inhibitor system and design its Fourier transform so that the linearized growth rate $\operatorname{Re}\lambda(k)$ is positive only near the comb $\{k \log p\}$ (a multi-band Turing instability). The ζ-weights of the amplitudes ($\frac{1}{k}p^{-k/2}$, or the von Mangoldt type) are not determined by linear theory and require a designed nonlinear saturation. For the log-periodicity, the system must moreover be mounted on the multiplicative variable $\xi = \log x$ — for instance, reaction–diffusion on an exponentially growing domain (Turing systems on growing domains have themselves been studied [20]) or scale-invariant transport $x\partial_x$.

7.3 Current status

To the compiler's knowledge, and within the scope of a literature search as of August 2026, no work constructing or realizing such a "ζ-like Turing system" can be found. This section is a blueprint, not a result.

7.4 Historical note

Morphogenesis and the computation of the zeros of ζ — the two subjects this note has placed side by side — have been treated together, in the literature, by no one but Turing himself. In the year after the morphogenesis paper [8], in one of his last papers [10] he reported numerical computations of the zeros of ζ and introduced a method for certifying the completeness of a list of zeros — known today as "Turing's method". It has remained an indispensable tool in all subsequent numerical verifications of the Riemann Hypothesis (for instance, up to height $3 \cdot 10^{12}$ [25]) and has been extended to the Selberg zeta function [23]. Thus the name "Turing" continues to appear in the literature on the zeros of ζ to this day; but the two subjects — morphogenesis and the zeros — sat side by side within one and the same person in 1952–53, and they were joined neither by him nor by anyone since.

8 Concluding remarks

"For which family of waves does the pattern become ζ-like?" — the answer is neither a single wavenumber nor a lattice, but the entire incommensurate comb of frequencies $\{k \log p\}$ guaranteed by unique factorization, together with its collective dual $\{\gamma_n\}$. The fundamental tone $W(t) = 2^{-1/2-it}$ of this ensemble carries the coordinate of the critical line engraved in its own intensity — as the amplitude $1/\sqrt{2}$ and the observed value $1/2$. But this is a rewording of the definition of the critical line, not an explanation of it.

It is my hope that, prompted by the geometric analysis above, someone will appear who discovers a new dynamics or a new theory.

A Computation scripts

All figures 1–11 of this note were produced by the single Python script below. Environment: Python 3.12 with `numpy`, `matplotlib`, `mpmath` (zeros at `mp.dps=12`; ζ values in the double-precision context `fp`) and `scipy`; runtime about 1–2 minutes. The zeros are computed by `mpmath.zetazero` and cached in `zeros100.txt`; the prime powers are enumerated by a sieve to 10^6. The label dictionary LBL makes the same code emit the Japanese and the English version of every figure.

```
# tone_figs.py --- all figures for "The Fundamental Tone 2^{-1/2-it}" (JA/EN)
# deps: numpy, matplotlib, mpmath, scipy.  run: python3 tone_figs.py  (~1-2 min)
import os, math, numpy as np, matplotlib
matplotlib.use('Agg')
import matplotlib.pyplot as plt
from mpmath import mp, fp, zetazero, siegeltheta, digamma
from scipy.optimize import brentq
mp.dps = 12
LOG = math.log; PI = math.pi; R2 = 2**-0.5   # R2 = 1/sqrt(2)

# ---------- zeros of zeta (cached in zeros100.txt) ----------
if os.path.exists('zeros100.txt'):
    G = [float(x) for x in open('zeros100.txt').read().split()]
else:
    G = [float(zetazero(n).imag) for n in range(1, 101)]
    open('zeros100.txt', 'w').write('\n'.join('%.15f' % g for g in G))
G = np.array(G)
TH = np.array([float(siegeltheta(g)) for g in G])          # theta(gamma_n)

# ---------- prime powers n = p^k <= 10^6, Lambda(n) = log p ----------
N = 10**6; sv = bytearray([1])*(N+1); sv[0] = sv[1] = 0
for i in range(2, 1001):
    if sv[i]: sv[i*i::i] = bytearray(len(sv[i*i::i]))
primes = [i for i in range(2, N+1) if sv[i]]
pp = []
for p in primes:
    q = p
    while q <= N: pp.append((q, math.log(p))); q *= p
pp.sort()
n_arr = np.array([a for a, _ in pp], float)
Lam   = np.array([b for _, b in pp])
w_vm  = Lam/np.sqrt(n_arr); ln = np.log(n_arr)             # von Mangoldt weights

def comb(max_om=3.45):                                     # prime comb (p,k,k log p)
    out = []
    for p in [2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31]:
        k = 1
        while k*LOG(p) <= max_om: out.append((p, k, k*LOG(p))); k += 1
    return sorted(out, key=lambda t: t[2])

def psiN(x, Nz):                                           # explicit formula, Nz zeros
    g = G[:Nz]; r = np.hypot(0.5, g); th = np.arctan2(g, 0.5)
    return (x - np.sum(2*np.sqrt(x)*np.cos(g*LOG(x)-th)/r)
            - LOG(2*PI) - 0.5*LOG(1-x**-2))

LBL = {
 'ja': dict(
   kc='臨界波数 $k_c$',
   grow='成長率 $\\mathrm{Re}\\,\\lambda$',
   turing='(a) 通常のチューリング不安定性',
   zlike='(b)「$\\zeta$ 的」多バンド設計（模式図）',
   gap='ギャップ $(0,\\log 2)$',
   stair='$\\psi(x)$（真の素数階段）',
   nz='零点 %d 個',
   xi='$\\xi=\\log x$',
   comb='零点波の和 $\\mathrm{e}^{\\xi/2}-2\\sum_{n\\leq100}\\cos(\\gamma_n\\xi)$',
   hist='単位平均化した隣接間隔（零点 1--100）',
   gue='GUE（Wigner 近似）',
   poi='ポアソン $\\mathrm{e}^{-s}$',
   land='$-\\sum_{n\\leq X}\\Lambda(n)n^{-1/2}\\cos(t\\log n)$',
   pi_w='$\\Pi$ 型振幅 $k^{-1}p^{-k/2}$',
   vm_w='von Mangoldt 振幅 $\\Lambda(n)/\\sqrt{n}$',
   seat='空席：$\\omega=\\gamma_E$ なら振幅 $0.7493$',
   fund='基音',
```

```
    strong7=' 最強は $n=7$',
    curve='$f(x)=(\\log x)/\\sqrt{x}$',
    mirr1='$|2^{-s}|=2^{-\\sigma}$',
    mirr2='$|2^{-(1-s)}|=2^{\\sigma-1}$',
    prod=' 積 $=1/2$（不変量）',
    bal=' 均衡点 $(1/2,\\,2^{-1/2})$',
    sub1='(i) 基音のみ：結晶',
    sub2='(ii) $2$ の塔：厳密周期（零点なし）',
    sub3='(iii) 二素数 $\\{2,3\\}$：準周期のうなり',
    sub4='(iv) 全合奏：$\\log|\\zeta(\\frac{1}{2}+it)|$, 零点が特異点として現れる',
   ),
  'en': dict(
    kc='critical wavenumber $k_c$',
    grow='growth rate $\\mathrm{Re}\\,\\lambda$',
    turing='(a) ordinary Turing instability',
    zlike="(b) ``$\\zeta$-like'' multi-band design (schematic)",
    gap='gap $(0,\\log 2)$',
    stair='$\\psi(x)$ (true prime staircase)',
    nz='%d zeros',
    xi='$\\xi=\\log x$',
    comb='sum of zero waves $\\mathrm{e}^{\\xi/2}-2\\sum_{n\\leq100}\\cos(\\gamma_n\\xi)$',
    hist='unfolded nearest-neighbour spacings (zeros 1--100)',
    gue='GUE (Wigner surmise)',
    poi='Poisson $\\mathrm{e}^{-s}$',
    land='$-\\sum_{n\\leq X}\\Lambda(n)n^{-1/2}\\cos(t\\log n)$',
    pi_w='$\\Pi$-type amplitude $k^{-1}p^{-k/2}$',
    vm_w='von Mangoldt amplitude $\\Lambda(n)/\\sqrt{n}$',
    seat='empty seat: at $\\omega=\\gamma_E$ amplitude would be $0.7493$',
    fund='fundamental',
    strong7='strongest is $n=7$',
    curve='$f(x)=(\\log x)/\\sqrt{x}$',
    mirr1='$|2^{-s}|=2^{-\\sigma}$',
    mirr2='$|2^{-(1-s)}|=2^{\\sigma-1}$',
    prod='product $=1/2$ (invariant)',
    bal='balance point $(1/2,\\,2^{-1/2})$',
    sub1='(i) fundamental alone: a crystal',
    sub2='(ii) tower of 2: exactly periodic (no zeros)',
    sub3='(iii) two primes $\\{2,3\\}$: quasiperiodic beats',
    sub4='(iv) full ensemble: $\\log|\\zeta(\\frac{1}{2}+it)|$, zeros as singularities',
   ),
}

for LANG in ['ja', 'en']:
    if LANG == 'ja':
        try: import japanize_matplotlib
        except Exception: pass
    L = LBL[LANG]; D = 'figs_tone_' + LANG; os.makedirs(D, exist_ok=True)
    plt.rcParams.update({'figure.dpi': 110, 'savefig.dpi': 170, 'font.size': 10})

    # fig1: dispersion relations (schematic)
    fig, (a, b) = plt.subplots(1, 2, figsize=(7.4, 2.9))
    k = np.linspace(0, 2.2, 400)
    a.plot(k, 0.25-(k**2-1)**2, 'b-'); a.axhline(0, color='k', lw=0.7)
    a.axvline(1, color='gray', ls=':'); a.set_title(L['turing'], fontsize=10)
    a.set_xlabel('$k$'); a.set_ylabel(L['grow']); a.set_ylim(-1, 0.5)
    a.annotate(L['kc'], xy=(1, 0.25), xytext=(1.25, 0.32), fontsize=9,
               arrowprops=dict(arrowstyle='->'))
    om = np.linspace(0, 3.4, 1200); lam = -0.15*np.ones_like(om)
    for p, kk, w in comb(3.4):
        lam += (p**(-kk/2)/kk)*1.1*np.exp(-(om-w)**2/(2*0.02**2))
    b.plot(om, lam, 'b-'); b.axhline(0, color='k', lw=0.7)
    b.axvspan(0, LOG(2), color='0.9'); b.set_title(L['zlike'], fontsize=10)
    b.set_xlabel('$\\omega$'); b.set_ylim(-0.3, 0.75)
    b.text(0.05, 0.55, L['gap'], fontsize=8)
    fig.tight_layout(); fig.savefig(D+'/fig1_dispersion.png'); plt.close(fig)

    # fig2: wave-packet form of the explicit formula (partial sums)
```

```
xs = np.linspace(2.02, 50, 1500)
xt = np.arange(2, 51); st = np.cumsum([Lam[np.searchsorted(n_arr, m)] if m in n_arr else 0
                                       for m in xt])
fig, ax = plt.subplots(2, 2, figsize=(7.4, 5.2), sharex=True, sharey=True)
for A, Nz in zip(ax.flat, [1, 3, 10, 100]):
    A.step(xt, st, where='post', color='k', lw=1.0, label=L['stair'])
    A.plot(xs, [psiN(x, Nz) for x in xs], 'r-', lw=0.9, label=L['nz'] % Nz)
    A.legend(fontsize=8, loc='upper left')
for A in ax[1]: A.set_xlabel('$x$')
fig.tight_layout(); fig.savefig(D+'/fig2_packet.png'); plt.close(fig)

# fig3: the d-psi side --- zero waves detect the prime powers
xi = np.arange(0.55, 2.72, 0.002)
Dxi = np.exp(xi/2) - 2*np.cos(np.outer(xi, G)).sum(1)
fig, ax = plt.subplots(figsize=(7.2, 3.4))
ax.plot(xi, Dxi, 'b-', lw=0.8, label=L['comb'])
for m in [2, 3, 4, 5, 7, 8, 9, 11, 13]:
    ax.axvline(LOG(m), color='r', ls=':', lw=0.8)
    ax.annotate('$%d$' % m, xy=(LOG(m), 0.93*Dxi.max()), fontsize=9, ha='center')
ax.set_xlabel(L['xi']); ax.legend(fontsize=9, loc='lower right')
fig.tight_layout(); fig.savefig(D+'/fig3_dpsi.png'); plt.close(fig)

# fig4: unfolded spacing histogram vs GUE / Poisson
s = np.diff(TH)/PI
fig, ax = plt.subplots(figsize=(6.2, 3.4))
ax.hist(s, bins=np.arange(0, 3.01, 0.25), density=True, color='0.8',
        edgecolor='k', label=L['hist'])
ss = np.linspace(0, 3, 300)
ax.plot(ss, (32/PI**2)*ss**2*np.exp(-4*ss**2/PI), 'r-', label=L['gue'])
ax.plot(ss, np.exp(-ss), 'g--', label=L['poi'])
ax.set_xlabel('$s$'); ax.legend(fontsize=9)
fig.tight_layout(); fig.savefig(D+'/fig4_gue.png'); plt.close(fig)

# fig5: the dual direction --- prime sums peak at the zeros (Landau)
t = np.arange(10, 42, 0.02)
fig, ax = plt.subplots(figsize=(7.2, 4.0))
for X, off, c in [(1e2, 0, 'g'), (1e4, 6, 'b'), (1e6, 14, 'r')]:
    m = n_arr <= X; w = w_vm[m]; l = ln[m]
    S = np.empty(len(t))
    for i in range(0, len(t), 100):
        S[i:i+100] = np.cos(np.outer(t[i:i+100], l)) @ w
    pole = np.real(math.sqrt(X)*np.exp(-1j*t*math.log(X))/(0.5-1j*t))
    ax.plot(t, pole-S+off, c+'-', lw=0.9,      # pole term removed
            label='$X=10^{%d}$' % round(math.log10(X)))
for g in G[G < 42]: ax.axvline(g, color='k', ls=':', lw=0.6)
ax.set_xlabel('$t$'); ax.set_ylabel(L['land'])
ax.legend(fontsize=9, loc='upper left')
fig.tight_layout(); fig.savefig(D+'/fig5_landau.png'); plt.close(fig)

# fig6: the two normalizations of the prime comb
fig, (a, b) = plt.subplots(2, 1, figsize=(7.0, 5.0), sharex=True)
for p, kk, w in comb():
    a.plot([w, w], [0, p**(-kk/2)/kk], 'b-', lw=2)
    b.plot([w, w], [0, LOG(p)*p**(-kk/2)], 'g-', lw=2)
for A in (a, b):
    A.axhline(R2, color='r', ls='--', lw=1); A.axvspan(0, LOG(2), color='0.92')
a.annotate(L['fund']+' $2$', xy=(LOG(2), R2), xytext=(0.9, 0.66), fontsize=9,
           arrowprops=dict(arrowstyle='->'))
a.plot(0.5772, 0.02, 'kv', ms=6)
a.annotate('$\\gamma_E$', xy=(0.5772, 0.05), ha='center', fontsize=9)
b.annotate(L['strong7'], xy=(LOG(7), LOG(7)*7**-0.5), xytext=(2.35, 0.68), fontsize=9,
           arrowprops=dict(arrowstyle='->'))
a.set_ylabel(L['pi_w']); b.set_ylabel(L['vm_w']); b.set_xlabel('$\\omega=k\\log p$')
a.text(0.06, 0.62, '$2^{-1/2}$', color='r', fontsize=9)
fig.tight_layout(); fig.savefig(D+'/fig6_spectra.png'); plt.close(fig)

# fig7: the transcendence picture for the von Mangoldt normalization
```

```
f = lambda x: LOG(x)/math.sqrt(x)
x1 = brentq(lambda x: f(x)-R2, 2.5, 7.0); x2 = brentq(lambda x: f(x)-R2, 8.0, 25.0)
xs2 = np.linspace(1.6, 30, 600)
fig, ax = plt.subplots(figsize=(7.0, 3.6))
ax.plot(xs2, [f(x) for x in xs2], 'b-', label=L['curve'])
ax.axhline(R2, color='r', ls='--')
for m in [2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29]:
    ax.plot(m, f(m), 'ko', ms=4)
ax.plot(math.e**2, 2/math.e, 'r*', ms=11)
ax.annotate('$x=\\mathrm{e}^2,\\ f=2/\\mathrm{e}=0.7358$',
            xy=(math.e**2, 2/math.e),
            xytext=(10.5, 0.77), fontsize=9, arrowprops=dict(arrowstyle='->'))
ax.annotate('$x_1=%.3f$' % x1, xy=(x1, R2), xytext=(1.8, 0.60), fontsize=9,
            arrowprops=dict(arrowstyle='->'))
ax.annotate('$x_2=%.3f$' % x2, xy=(x2, R2), xytext=(16.5, 0.60), fontsize=9,
            arrowprops=dict(arrowstyle='->'))
ax.set_xlabel('$x$'); ax.set_ylim(0.3, 0.85); ax.legend(fontsize=9, loc='lower right')
fig.tight_layout(); fig.savefig(D+'/fig7_transcend.png'); plt.close(fig)

# fig8: the empty seat below the fundamental
fig, ax = plt.subplots(figsize=(6.4, 3.2))
ax.axvspan(0, LOG(2), color='0.92')
ax.plot([LOG(2), LOG(2)], [0, R2], 'b-', lw=3)
ax.plot([LOG(3), LOG(3)], [0, 3**-0.5], 'b-', lw=3)
ax.plot([0.5772, 0.5772], [0, math.exp(-0.5772/2)], 'r--', lw=2)
ax.plot(0.5772, math.exp(-0.5772/2), 'rx', ms=10, mew=2)
ax.annotate(L['seat'], xy=(0.5772, 0.75), xytext=(0.08, 0.86), fontsize=9,
            arrowprops=dict(arrowstyle='->'))
ax.annotate(L['fund']+' $2$：$(\\log 2,\\,2^{-1/2})$' if LANG == 'ja'
            else L['fund']+' $2$: $(\\log 2,\\,2^{-1/2})$',
            xy=(LOG(2), R2), xytext=(0.78, 0.78), fontsize=9,
            arrowprops=dict(arrowstyle='->'))
ax.annotate('$3$', xy=(LOG(3), 3**-0.5+0.02), ha='center', fontsize=10)
ax.set_xlim(0, 1.25); ax.set_ylim(0, 1.0); ax.set_xlabel('$\\omega$')
fig.tight_layout(); fig.savefig(D+'/fig8_seat.png'); plt.close(fig)

# fig9: the digamma identity at the center of the line
xs3 = np.linspace(0.07, 3, 400); dg = [float(digamma(x)) for x in xs3]
fig, ax = plt.subplots(figsize=(6.4, 3.4))
ax.plot(xs3, dg, 'b-'); ax.axhline(0, color='k', lw=0.6)
ax.plot(0.5, -0.5772-2*LOG(2), 'ro', ms=6)
ax.plot([0.5, 0.5], [0, -0.5772], 'g-', lw=3)
ax.plot([0.5, 0.5], [-0.5772, -0.5772-2*LOG(2)], 'm-', lw=3)
ax.annotate('$-\\gamma_E$', xy=(0.53, -0.30), color='g', fontsize=10)
ax.annotate('$-2\\log 2$', xy=(0.53, -1.25), color='m', fontsize=10)
ax.annotate('$\\psi_\\Gamma(1/2)=-1.9635$', xy=(0.5, -1.9635), xytext=(1.0, -2.6),
            fontsize=10, arrowprops=dict(arrowstyle='->'))
ax.set_xlabel('$x$'); ax.set_ylabel('$\\psi_\\Gamma(x)$'); ax.set_ylim(-4, 2)
fig.tight_layout(); fig.savefig(D+'/fig9_digamma.png'); plt.close(fig)

# fig10: the mirror-pair invariant and its balance point
sg = np.linspace(-0.5, 1.5, 400)
fig, (a, b) = plt.subplots(1, 2, figsize=(7.4, 3.2))
a.plot(sg, 2.0**(-sg), 'b-', label=L['mirr1'])
a.plot(sg, 2.0**(sg-1), 'g-', label=L['mirr2'])
a.axhline(0.5, color='r', ls='--', label=L['prod'])
a.plot(0.5, R2, 'ko', ms=6); a.axvline(0.5, color='k', ls=':', lw=0.8)
a.annotate(L['bal'], xy=(0.5, R2), xytext=(0.62, 1.35), fontsize=9,
           arrowprops=dict(arrowstyle='->'))
a.set_xlabel('$\\sigma$'); a.legend(fontsize=8, loc='upper center')
for p, c in [(2, 'r'), (3, 'b'), (5, 'g')]:
    b.plot(sg, (p*1.0)**(-sg)*(p*1.0)**(sg-1)*np.ones_like(sg), c+'--', lw=1)
    b.plot(0.5, 1.0/p, c+'o', ms=5)
    b.annotate('$1/%d$' % p, xy=(1.35, 1.0/p+0.015), color=c, fontsize=9)
b.axhline(0.5, color='r', lw=0.5)
b.set_xlabel('$\\sigma$'); b.set_ylim(0, 0.65)
b.set_title('$|p^{-s}||p^{-(1-s)}|=1/p$', fontsize=10)
```

```
fig.tight_layout(); fig.savefig(D+'/fig10_mirror.png'); plt.close(fig)

# fig11: subfamilies (i)-(iv) of Section 7.1
tt = np.arange(0.2, 50, 0.02)
z2 = R2*np.exp(-1j*tt*LOG(2)); z3 = (3**-0.5)*np.exp(-1j*tt*LOG(3))
rows = [np.sqrt(2)*np.cos(tt*LOG(2)), (-np.log(1-z2)).real,
        (-np.log(1-z2)-np.log(1-z3)).real,
        np.array([math.log(abs(fp.zeta(complex(0.5, x)))) for x in tt])]
fig, ax = plt.subplots(4, 1, figsize=(7.2, 7.6), sharex=True)
for A, y, ttl in zip(ax, rows, [L['sub1'], L['sub2'], L['sub3'], L['sub4']]):
    A.plot(tt, y, 'b-', lw=0.8); A.set_title(ttl, fontsize=10)
for g in G[G < 50]: ax[3].axvline(g, color='r', ls=':', lw=0.7)
ax[3].set_xlabel('$t$'); ax[3].set_ylim(-6, 2)
fig.tight_layout(); fig.savefig(D+'/fig11_subfam.png'); plt.close(fig)

print(LANG, 'done:', len(os.listdir(D)), 'figures')
```

References

[1] L. Euler, *Variae observationes circa series infinitas*, Comment. Acad. Sci. Petrop. **9** (1737/1744), 160–188.

[2] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsber. Königl. Preuss. Akad. Wiss. Berlin (1859), 671–680.

[3] F. Mertens, *Ein Beitrag zur analytischen Zahlentheorie*, J. Reine Angew. Math. **78** (1874), 46–62.

[4] F. Lindemann, *Über die Zahl* π, Math. Ann. **20** (1882), 213–225.

[5] H. von Mangoldt, *Zu Riemanns Abhandlung "Über die Anzahl der Primzahlen unter einer gegebenen Grösse"*, J. Reine Angew. Math. **114** (1895), 255–305.

[6] H. Bohr, *Almost Periodic Functions*, Chelsea, New York, 1947.

[7] B. Nyman, *On the One-Dimensional Translation Group and Semi-Group in Certain Function Spaces*, Thesis, Uppsala, 1950.

[8] A. M. Turing, *The chemical basis of morphogenesis*, Phil. Trans. R. Soc. Lond. B **237** (1952), 37–72.

[9] A. Weil, *Sur les "formules explicites" de la théorie des nombres premiers*, Comm. Sém. Math. Univ. Lund, Tome Supplémentaire (1952), 252–265.

[10] A. M. Turing, *Some calculations of the Riemann zeta-function*, Proc. London Math. Soc. (3) **3** (1953), 99–117.

[11] A. Beurling, *A closure problem related to the Riemann zeta-function*, Proc. Nat. Acad. Sci. USA **41** (1955), 312–314.

[12] H. L. Montgomery, *The pair correlation of zeros of the zeta function*, Proc. Sympos. Pure Math. **24**, Amer. Math. Soc. (1973), 181–193.

[13] H. M. Edwards, *Riemann's Zeta Function*, Academic Press, New York, 1974.

[14] M. V. Berry, *Riemann's zeta function: a model for quantum chaos?*, in *Quantum Chaos and Statistical Nuclear Physics*, Lecture Notes in Physics **263**, Springer (1986), 1–17.

[15] A. M. Odlyzko, *On the distribution of spacings between zeros of the zeta function*, Math. Comp. **48** (1987), 273–308.

[16] M. C. Cross and P. C. Hohenberg, *Pattern formation outside of equilibrium*, Rev. Mod. Phys. **65** (1993), 851–1112.

[17] X.-J. Li, *The positivity of a sequence of numbers and the Riemann hypothesis*, J. Number Theory **65** (1997), 325–333.

[18] M. V. Berry and J. P. Keating, *The Riemann zeros and eigenvalue asymptotics*, SIAM Review **41** (1999), 236–266.

[19] A. Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*, Selecta Math. (N.S.) **5** (1999), 29–106.

[20] E. J. Crampin, E. A. Gaffney, and P. K. Maini, *Reaction and diffusion on growing domains: scenarios for robust pattern formation*, Bull. Math. Biol. **61** (1999), 1093–1120.

[21] J. C. Lagarias, *An elementary problem equivalent to the Riemann hypothesis*, Amer. Math. Monthly **109** (2002), 534–543.

[22] F. Dyson, *Birds and frogs*, Notices Amer. Math. Soc. **56** (2009), 212–223.

[23] A. R. Booker and D. J. Platt, *Turing's method for the Selberg zeta-function*, arXiv:1710.00603.

[24] S. Torquato, G. Zhang, and M. de Courcy-Ireland, *Uncovering multiscale order in the prime numbers via scattering*, J. Stat. Mech. (2018), 093401.

[25] D. Platt and T. Trudgian, *The Riemann hypothesis is true up to* $3 \cdot 10^{12}$, Bull. London Math. Soc. **53** (2021), 792–797.