

The Ponyos of the Riemann Zeta Function: the Winding–Undulation Decomposition of the Phase and a Spectroscopy of the Prime Comb

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August 19, 2026

Abstract

We define the *ponyos* of ζ as the two-dimensional colour bands of the phase $\arg \zeta(s)$ — the bands between level curves, the level curves themselves forming the *comb* that bounds and foliates them — and organize a dialogue about them into six parts. (1) An exact equation for the comb curves — the edges and centrelines of the ponyos — follows from the functional equation, and Stirling’s formula yields the asymptotic comb equation $-2\theta(t) + [\sigma(\sigma - 1)/2 + 1/8]/t + U(\sigma, t) \equiv c \pmod{2\pi}$, where θ is the Riemann–Siegel theta function and $U(\sigma, t) = \sum_{p,k} k^{-1} p^{k(\sigma-1)} \sin(kt \log p)$ is a superposition of prime waves. The exact equation is an identity, so its check (10^{-20}) verifies the implementation; the asymptotic form is tested against its predicted $1/t$ correction. The ponyo itself is the band between two comb curves: its width is $\Delta c/|\zeta'/\zeta|$ to first order, and its tip at a simple zero is a wedge of opening exactly Δc , rotated by $\arg \zeta'(\rho)$. (2) The anatomy of the ponyos — spacing, tilt, undulation, and the grounding of one ponyo at one nontrivial zero — is computed and measured; the orientation of the grounding is governed by $\arg \zeta'(\rho_n) = \arg(-i e^{-i\theta(\gamma_n)} Z'(\gamma_n))$, whose Gram-law drift makes the observed cyan-to-red pattern a typical rule rather than a theorem, while for ξ the alternation $\pm 90^\circ$ is exact. (3) The two terms of the ponyo phase are the two sides of the Riemann–Weil

*This note records, in article form, a dialogue held on August 17, 2026 (a user and Claude, Anthropic), compiled into paper form by Claude on the same day. It is a companion to the note *The Fundamental Tone $2^{-1/2-it}$ on the Critical Line and the Prime–Zero Duality* [32], produced from a parallel dialogue on the same day. The terms *wave tongue* (renamed *ponyo* in the present version) and *vertical grounding* translate coinages of the interlocutor, recorded in phase-portrait memoranda of August 15–16, 2026 [33]. As stated in Section 9, this note claims no new mathematical results: the derivations are elementary consequences of classical facts, and the numerical experiments, while to the compiler’s knowledge not in the literature in this form, establish no theorem. This is a revised version (v2), correcting issues identified in a follow-up review within the same dialogue: the overstated status of the spectroscopy experiment (Section 8); the omission of Hardy’s theorem (Section 6); the partially double-counted “two protections” argument; references previously uncited in the text (Speiser, Turing, Arias de Reyna, Dyson, Platt–Trudgian) now wired in; the spectral-gap scan extended to $\omega = 0.10$ and the spectroscopy repeated at three abscissae; uncertainty statements for the pitchfork exponents; figure and float layout defects; and an added Limitations subsection. Bibliographic details were written from memory during the dialogue and have not been checked against the sources; no systematic literature search was performed; references [32, 33] are non-archival. All errors are the compiler’s. In the present version (August 18, 2026) the author line has been changed, and the term *wave tongue* has been renamed *ponyo*. In the version of August 19, 2026, following a further remark by the author — that the ponyos and the comb of level curves fill the complex plane complementarily, and that equations (4) and (7) are, strictly, equations of the comb — Section 3.3 was added: it states the ponyo’s own equation as a band, derives its width laws and its wedge-shaped tip, and verifies them numerically; the definition in Section 3.1, the abstract, Figure 1 and the status table were adjusted accordingly.

explicit formula: the archimedean winding and the prime undulation; the identity is checked numerically to 2.6×10^{-14} . (4) The line $\sigma = 1/2$ is the unique place where the two mechanisms protecting ζ against vanishing (not fully independent; see Section 6) — the deterministic drift $(1/2 - \sigma) \log(t/2\pi)$ of $\log|\zeta|$ and the convergence of the prime-noise variance $\frac{1}{2} \sum_p p^{-2\sigma}$ — fail simultaneously, the latter with the slowest possible (Mertens $\log \log$) divergence. (5) The ponyo tips admit a precise symmetry-breaking reading through the de Bruijn–Newman flow: with the theorems $\text{RH} \Leftrightarrow \Lambda \leq 0$ and $\Lambda \geq 0$ (Rodgers–Tao), the Riemann Hypothesis is the statement $\Lambda = 0$ that ζ sits exactly at the phase boundary; a finite- N simulation on the actual zeros exhibits the mean-field pitchfork with exponents $\approx 1/2$, and the trivial zeros are identified as the frozen, fluctuation-free lattice of the archimedean factor. (6) A *spectroscopy of the ponyos* at the three abscissae $\sigma = -1, -2, -4$ reproduces the nine comb amplitudes $k^{-1}p^{k(\sigma-1)}$ to four–five digits; the fundamental amplitudes $0.25000/0.12500/0.03125$ follow the law $2^{\sigma-1}$, and the spectral gap below $\log 2$ is empty down to the resolution limit (≈ 0.06); the fundamental tone $W(t) = 2^{-1/2-it}$ of the companion note [32] is the strongest line. As Section 8 makes explicit, this spectroscopy is an implementation check of an exact identity, not a falsifiable experiment. A closing table records the status — classical theorem, elementary derivation, exact identity, numerical verification, illustration, or open — of every claim. This note claims no new mathematical results.

1 Introduction

Phase portraits of the Riemann zeta function — colorings of the s -plane by $\arg \zeta(s)$ [28] — show, to the left of the critical strip, a family of nearly horizontal colored bands that stretch in from $\sigma = -\infty$, undulate gently, and terminate on the critical strip, where all colors meet at the nontrivial zeros. In memoranda of August 15–16, 2026 [33], the interlocutor of the present dialogue named these bands *ponyos* (*wave tongues*) and the perpendicular manner of their termination at zeros *vertical grounding*, and asked what the ponyos are and why they behave as observed. This note is the compiled answer.

The organizing principle that emerged is a two-term decomposition of the phase. Along every vertical line the phase of ζ consists of a *winding*, a smooth deterministic rotation supplied by the archimedean factor $\pi^{-s/2}\Gamma(s/2)$ of the completed zeta function, and an *undulation*, a superposition of prime waves $k^{-1}p^{k(\sigma-1)} \sin(kt \log p)$ supplied by the Euler product through the functional equation. The winding fixes the number and mean spacing of the ponyos; the undulation bends them; the ponyos ground, one each, at the nontrivial zeros; and the two terms are precisely the two sides of the Riemann–Weil explicit formula. Every phenomenon the interlocutor had recorded — the near-constancy of the ponyo spacing, the uniform cyan-to-red orientation of the grounding for ζ against the alternating green–blue orientation for ξ , the regular alternation at the trivial zeros — is computed below from this decomposition, together with the precise sense in which each observation is a theorem, a typical rule, or a restatement.

Section 3 derives the exact and asymptotic equations of the comb curves, then the equation of the ponyo band itself — its width laws and its wedge-shaped tip — and verifies them. Section 4 computes the anatomy: spacing, tilt, coherence, grounding, counting, and orientation. Section 5 identifies the two terms with the two sides of the explicit formula and checks the identity numerically. Section 6 computes why the critical line is critical. Section 7 gives the symmetry-breaking reading of the ponyo tips through the de Bruijn–Newman flow, and places the trivial zeros. Section 8 reports the spectroscopy of the ponyos — with its scientific status stated explicitly — which reads the prime comb, including the fundamental tone of the companion note [32], directly in the ponyo displacement field. Section 9 tabulates the status of every claim; Section 10 concludes.

All numerical values reported below were produced during the dialogue in Python (`mpmath`,

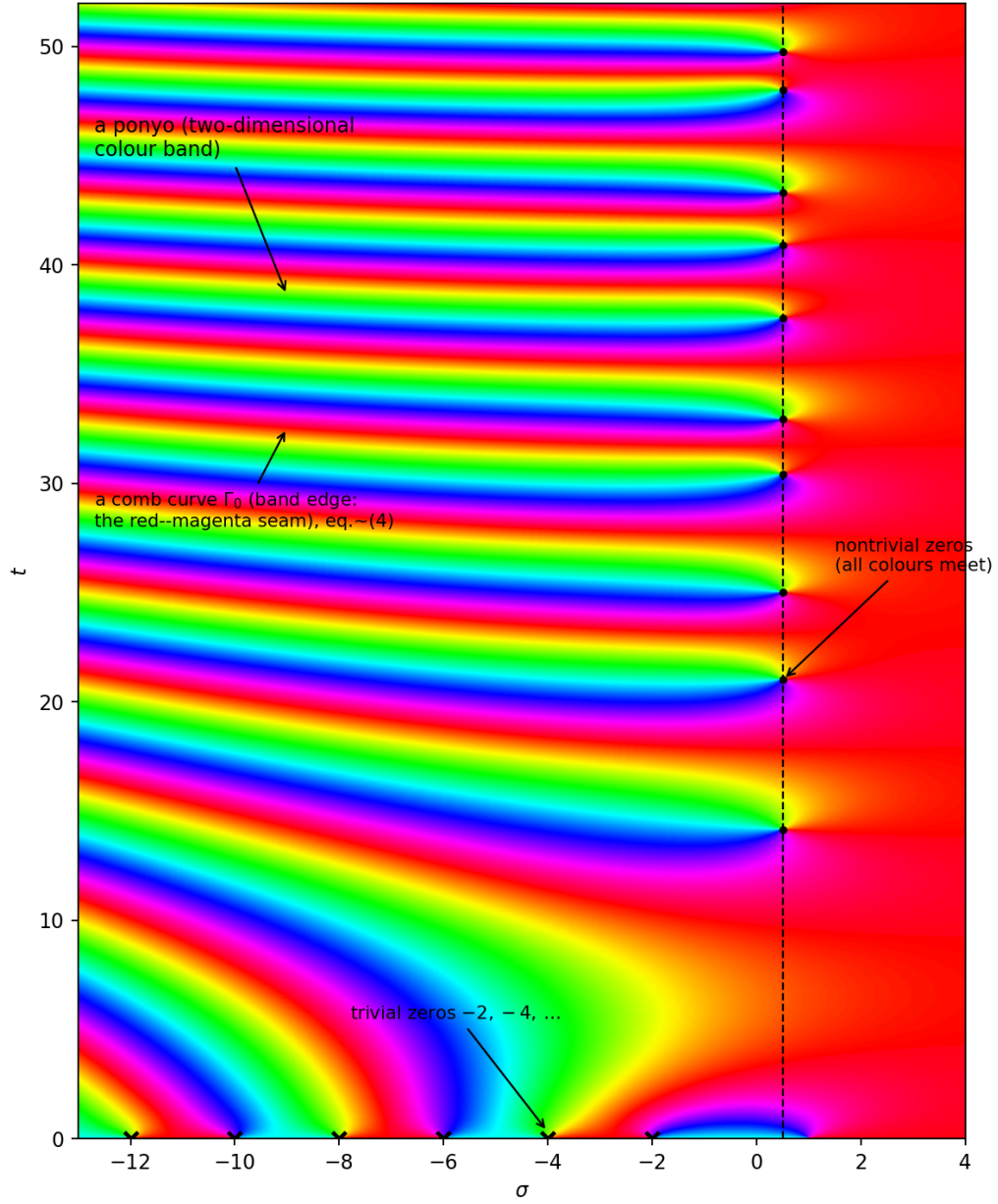


Figure 1: Phase portrait of $\arg \zeta(\sigma + it)$ (hue = phase; the exact anchors are red = 0 and cyan = π , with the yellow-green and blue-violet bands corresponding approximately to $\pi/2$ and $3\pi/2$). The ponyos stream in from the left and ground on the critical line (dashed); trivial zeros are marked on the real axis. One two-dimensional band (a ponyo) and one comb curve (a band edge) are annotated; see Section 3.3.

working precision 8–25 digits; `numpy`), and are quoted as computed. Tabulated and in-text values were obtained in arbitrary precision (`mpmath`); the figures were computed separately in double precision (`mpmath.fp`).

2 Notation and classical background

We write $s = \sigma + it$, and $\rho = \beta + i\gamma$ for a nontrivial zero; $\gamma_1 = 14.134725\dots$, $\gamma_2 = 21.022040\dots$. The completed function is $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, with $\xi(s) = \xi(1-s)$ [1]. The functional equation for ζ is $\zeta(s) = \chi(s)\zeta(1-s)$ with

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s), \quad |\chi(\sigma + it)| = \left(\frac{t}{2\pi}\right)^{1/2-\sigma} (1 + O(1/t)). \quad (1)$$

The Riemann–Siegel theta function is $\theta(t) = \arg \Gamma(1/4 + it/2) - \frac{t}{2} \log \pi$, so that $Z(t) = e^{i\theta(t)} \zeta(1/2 + it)$ is real and $\chi(1/2 + it) = e^{-2i\theta(t)}$; asymptotically $\theta(t) = \frac{t}{2} \log \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O(1/t)$. The zero-counting function satisfies the Riemann–von Mangoldt formula

$$N(T) = \frac{\theta(T)}{\pi} + 1 + S(T), \quad S(T) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + iT\right), \quad (2)$$

S being the fluctuation term, of typical size $\sqrt{\log \log T}$ (Selberg [9]). The explicit formula of Riemann [1] and von Mangoldt [3] reads, for $x > 1$ not a prime power,

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log 2\pi - \frac{1}{2} \log(1 - x^{-2}), \quad (3)$$

the sum over zeros taken symmetrically; Weil [11] recast it as a distributional identity recalled in Section 5. For the classical material see Edwards [15] and Titchmarsh [18].

3 The equations of the ponys

3.1 Definition and exact form

A *ponyo* is, properly, a two-dimensional object: a connected component of a colour band $\{s : c_1 < \arg \zeta(s) < c_2\}$. The one-dimensional level curves $\Gamma_c = \{s : \arg \zeta(s) = c\}$ we call *comb curves*, and their family the *comb*: they are the edges and centrelines of the ponys, and — as the author observes — the ponys and the comb fill the complex plane complementarily (Section 3.3). The centreline of the cyan ponyo of a standard phase portrait is Γ_{π} , i.e. the locus $\{\operatorname{Im} \zeta = 0, \operatorname{Re} \zeta < 0\}$. Equations (4)–(8) below are, strictly, equations of the comb curves; the ponyo’s own equation is given in Section 3.3. The loci $\{\operatorname{Im} \zeta = 0\}$ and $\{\operatorname{Re} \zeta = 0\}$ are precisely the curves studied in Arias de Reyna’s “X-ray” of ζ [26]; the ponyo analysis below very likely overlaps with that circle of ideas, and no systematic comparison with the literature was carried out during the dialogue. For $\sigma < 0$ the functional equation gives the exact comb equation

$$t \log 2\pi + \arg \sin\left(\frac{\pi s}{2}\right) + \arg \Gamma(1-s) + \arg \zeta(1-s) \equiv c \pmod{2\pi}, \quad (4)$$

with $\arg \sin(\pi s/2) = \operatorname{atan2}(\cos \frac{\pi \sigma}{2} \sinh \frac{\pi t}{2}, \sin \frac{\pi \sigma}{2} \cosh \frac{\pi t}{2})$. Equation (4) was verified at the sample points $s = (-3, 40), (-1, 25), (-5, 7), (-2, 0.8)$ with residuals of order 10^{-20} at working precision 20. As $t \rightarrow 0$ the $\operatorname{atan2}$ term degenerates to 0 or π according to the sign of $\sin(\pi \sigma/2)$, which flips at each even integer: this reproduces the interlocutor’s observation that the color pairs at the trivial zeros $-2, -4, -6, \dots$ alternate.

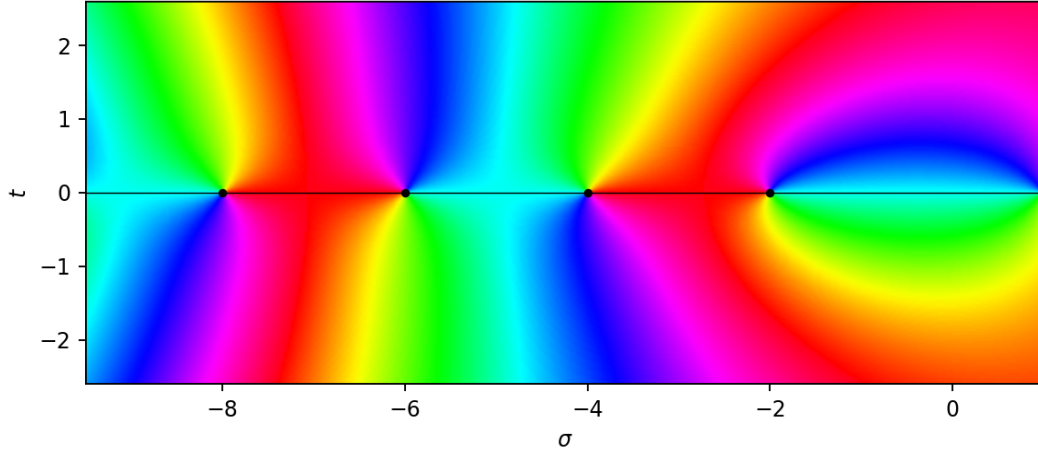


Figure 2: Neighbourhood of the trivial zeros: the color pairs alternate at $-2, -4, -6, \dots$, as forced by the sign of $\sin(\pi\sigma/2)$ in (4).

3.2 Asymptotic form

Stirling's series applied to $\chi = \pi^{s-1/2} \Gamma(\frac{1-s}{2})/\Gamma(\frac{s}{2})$ gives, for bounded σ and $t \rightarrow \infty$,

$$\arg \chi(\sigma + it) = -2\theta(t) + \frac{1}{t} \left(\frac{\sigma(\sigma-1)}{2} + \frac{1}{8} \right) + O\left(\frac{1+\sigma^4}{t^3}\right), \quad (5)$$

which passes the consistency check $\arg \chi = -2\theta$ exactly at $\sigma = 1/2$. Writing $w = 1 - s$ and expanding $\log \zeta(w) = \sum_{p,k} k^{-1} p^{-kw}$ (absolutely convergent since $\operatorname{Re} w = 1 - \sigma > 1$),

$$\arg \zeta(1-s) = U(\sigma, t) := \sum_p \sum_{k \geq 1} \frac{1}{k} p^{k(\sigma-1)} \sin(kt \log p), \quad (6)$$

a superposition of *prime waves* of frequency $k \log p$ and amplitude $k^{-1} p^{k(\sigma-1)}$ — the general- σ version of the mode table of [32]. The asymptotic comb equation is therefore

$$\boxed{-2\theta(t) + \frac{1}{t} \left(\frac{\sigma(\sigma-1)}{2} + \frac{1}{8} \right) + U(\sigma, t) \equiv c \pmod{2\pi}.} \quad (7)$$

Numerically, the residual $\arg \zeta - (-2\theta + U)$ matches the $1/t$ correction of (5): at $(\sigma, t) = (-3, 40)$ the measured residual is $+0.1542$ against the predicted $6.125/40 = 0.1531$; at $(-5, 60)$, $+0.2516$ against $15.125/60 = 0.2521$; at $(-2, 100)$, $+0.0281$ against 0.0313 (the difference being the truncated prime tail). At $\sigma = 0$ the series (6) is only conditionally summable and the truncated comparison degrades, as expected.

Solving (7) locally as a graph $t(\sigma)$ about an anchor height t_c with $-2\theta(t_c) \equiv c$,

$$t(\sigma) \approx t_c + \frac{1}{L} \left[\frac{\sigma(\sigma-1)/2 + 1/8}{t_c} + U(\sigma, t_c) \right], \quad L = \log \frac{t_c}{2\pi}, \quad (8)$$

i.e. a horizontal line, tilted parabolically to the upper left by the archimedean correction, and undulated by the prime waves.

Equations (4)–(8) describe the comb — the one-dimensional skeleton. The ponyo itself is two-dimensional; we turn to its own equation.

3.3 The ponyo itself: the band, its width, and its tip

Fix a hue interval (c_1, c_2) of width $\Delta c = c_2 - c_1$ and write $\bar{c} = (c_1 + c_2)/2$. The exact equation of the ponyo is the double inequality

$$c_1 < \arg \zeta(s) < c_2 \pmod{2\pi}, \quad (9)$$

a connected component thereof. Its boundary consists of the two comb curves $\Gamma_{c_1}, \Gamma_{c_2}$, and its interior is foliated by the comb curves of the intermediate hues, $P = \bigcup_{c_1 < c < c_2} \Gamma_c$: the ponyo is the comb thickened over an interval of hues, and the comb curves are ponyos of zero width. Fixing any finite partition $0 = c_0 < c_1 < \dots < c_K = 2\pi$ of the colour circle, the plane minus the zeros and the pole decomposes as the disjoint union of the open ponyos and the closed comb $\bigcup_j \Gamma_{c_j}$: the two objects fill the plane complementarily. Asymptotically, by (7), the ponyo is the band

$$\left| -2\theta(t) + \frac{1}{t} \left(\frac{\sigma(\sigma-1)}{2} + \frac{1}{8} \right) + U(\sigma, t) - \bar{c} \right| < \frac{\Delta c}{2} \pmod{2\pi}, \quad (10)$$

whose two edges are the graphs (8) with $c = c_1$ and $c = c_2$.

Width. Two laws follow at once. Vertically, differencing (8) in c — equivalently, dividing Δc by $|\partial_t \arg \zeta| = |\operatorname{Re}(\zeta'/\zeta)| \approx 2\theta' - U'$ — gives the thickness

$$w_t(\sigma, t) = \frac{\Delta c}{2\theta'(t) - U'(\sigma, t)} = \frac{\Delta c}{2\pi} \Delta t, \quad (11)$$

with Δt the comb spacing (13): a hue interval occupying the fraction $\Delta c/2\pi$ of the colour circle occupies the same fraction of the vertical period. Perpendicular to the band, since $\arg \zeta = \operatorname{Im} \log \zeta$ has $|\nabla \arg \zeta| = |\zeta'/\zeta|$, the width is

$$w_\perp(s) = \frac{\Delta c}{|\zeta'/\zeta(s)|} + O(\Delta c^2), \quad (12)$$

an exact local law wherever $\zeta' \neq 0$. Both were checked for the cyan ponyo ($\bar{c} = \pi$, $\Delta c = \pi/2$): at $s = -2 + 40.6i$, $|\zeta'/\zeta| = 1.92892$ predicts $w_\perp = 0.81434$ against a measured 0.81073 (the -0.4% being the $O(\Delta c^2)$ term); at $\sigma = -2$ the vertical thickness of the band that grounds at γ_7 is measured as 0.81087 against (11) = 0.81104 (-0.02%).

Tip. Near a zero ρ of multiplicity m one has $\arg \zeta(s) = \arg a_m + m \arg(s - \rho) + O(|s - \rho|)$, with a_m the leading Taylor coefficient, so (9) becomes $\arg(s - \rho) \in ((c_1 - \arg a_m)/m, (c_2 - \arg a_m)/m)$ modulo $2\pi/m$: the ponyo ends in a wedge (for $m = 1$, a single wedge) of opening *exactly* $\Delta c/m$, with bisector direction $\bar{c} - \arg \zeta'(\rho)$ in the simple case. This is the tapering “tongue tip” of the portraits, now with both its angle and its orientation: the width $\Delta c |s - \rho|$ closes linearly, in agreement with (12) since $|\zeta'/\zeta| \sim 1/|s - \rho|$; and by the orientation law (14), with $\arg \zeta'(\rho_n)$ typically near 0° (Section 4.3), the cyan wedge ($\bar{c} = \pi$) opens leftward and the red wedge rightward — the observed geometry of the grounding. Measured at ρ_7 (where $\arg \zeta'(\rho_7) = -11.7^\circ$): the angular measure of the band $\Delta c = 90^\circ$ on circles of radius $r = 0.10, 0.03, 0.01$ about ρ_7 is $82.7^\circ, 87.7^\circ, 89.3^\circ$, converging linearly to 90° . Figure 3 shows the band, its comb edges, and the wedge. At the pole $s = 1$ the same expansion holds with $m = -1$: every ponyo also has one wedge there, of the same opening but with the colour order reversed — the unique “anti-grounding” of the plane.

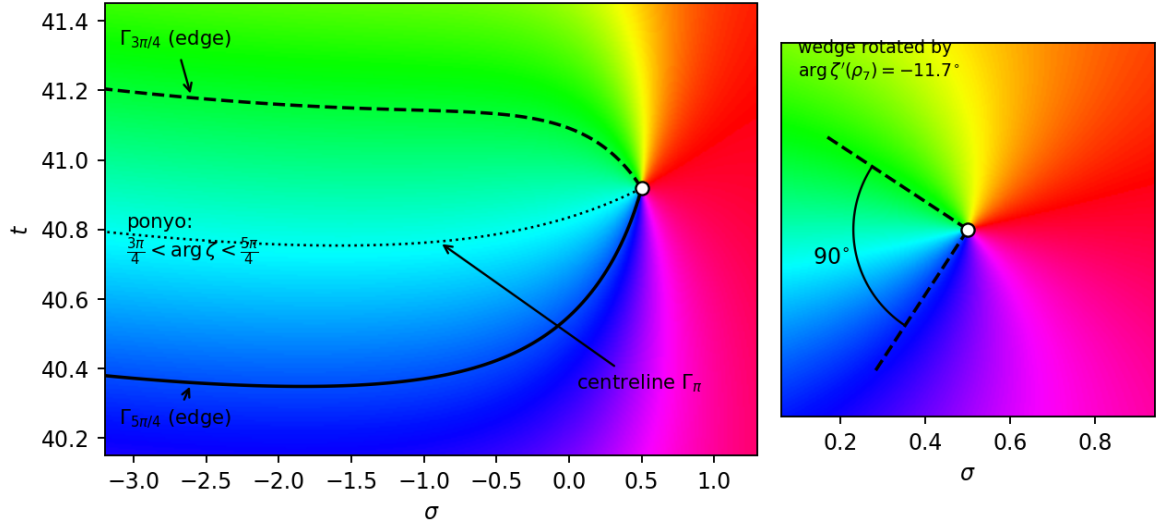


Figure 3: The ponyo itself, for the cyan band $\bar{c} = \pi$, $\Delta c = \pi/2$, near its grounding at γ_7 . Left: the band (9) with its two comb-curve edges $\Gamma_{3\pi/4}, \Gamma_{5\pi/4}$ (solid black) and its centreline Γ_π (dotted); axes not to equal scale. Right: equal-scale close-up of the tip — a wedge of opening exactly $\Delta c = 90^\circ$, rotated by $\arg \zeta'(\rho_7) = -11.7^\circ$ (dashed), as predicted by (14).

3.4 Vertical grounding is a local property

The interlocutor's original conjecture identified the set $\Gamma = \{s : \operatorname{Re}(\zeta'/\zeta)(s) = 0\}$ ("vertical grounding of the phase in the imaginary direction") with the critical line $L = \{\operatorname{Re} s = 1/2\}$, and proposed the chain $Z_{\text{nt}} \subseteq \Gamma \subseteq L$. The dialogue began by testing this. Near any zero ρ of multiplicity m one has $\zeta'/\zeta \sim m/(s-\rho)$, so $\operatorname{Re}(\zeta'/\zeta) = 0$ is locally the vertical line $\sigma = \beta$ *wherever* ρ *lies*: vertical grounding is an automatic local property of zeros and carries no information about their abscissae. Moreover $\Gamma \subseteq L$ is false for ζ : on the line itself $\operatorname{Re}(\zeta'/\zeta)(\frac{1}{2} + 20i) = -0.579 \neq 0$; the point $(\sigma, t) = (0.5041, 14.2347)$, inside the strip and adjacent to the first zero, lies on Γ off the line; and the first complex zero of ζ' , at $s = 2.46316 + 23.29832i$ (where $\zeta \neq 0$), lies on Γ with $\sigma \approx 2.46$. For ξ , by contrast, $\operatorname{Re}(\xi'/\xi) = 0$ holds identically on the line, and the statement that it holds *only* there is equivalent to the Riemann Hypothesis (Lagarias [24]); the chain thus relocates, rather than reduces, the difficulty. Classically, the geometry of such loci near the strip goes back to Speiser [8], whose theorem — RH is equivalent to $\zeta' \neq 0$ in $0 < \sigma < 1/2$ — shows that statements about ζ' -loci can carry the full weight of RH. These findings motivated the global analysis of the ponyos that follows.

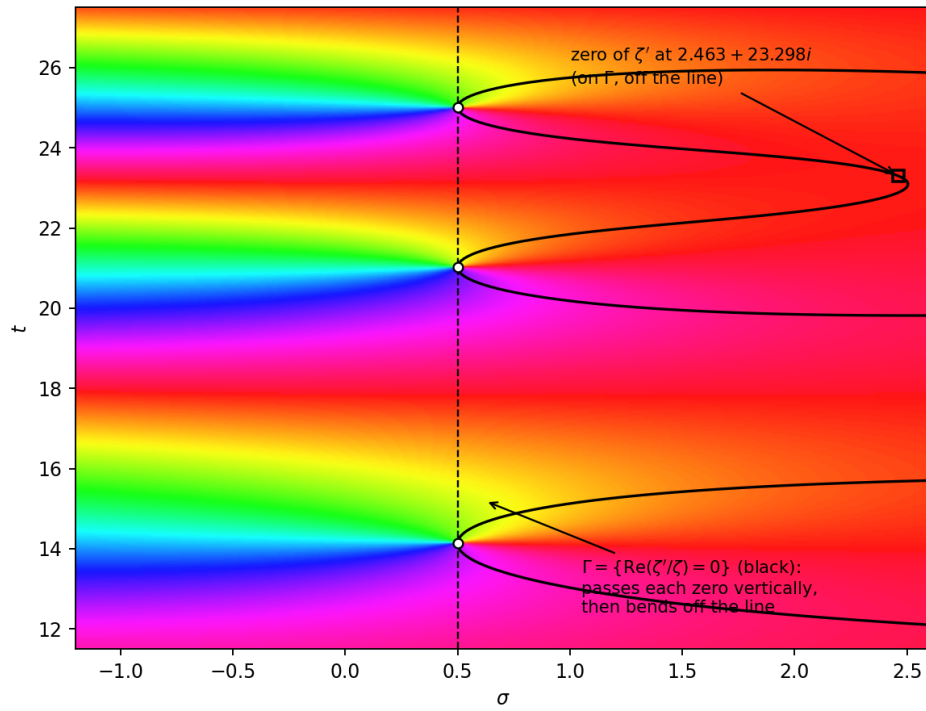


Figure 4: Grounding close-up with the curve $\Gamma = \{\operatorname{Re}(\zeta'/\zeta) = 0\}$ (black). Γ passes through each zero vertically — a local, position-independent property — and then bends off the line; the boxed marker is the first complex zero of ζ' , on Γ at $\sigma \approx 2.46$.

4 Anatomy of the ponys

4.1 Spacing, and why it looks constant

Consecutive ponys of the same color advance the phase (7) by 2π , so (by the mean value theorem, with the derivative evaluated at an interior point)

$$\Delta t = \frac{2\pi}{2\theta'(t) - U'(\sigma, t)} = \frac{2\pi}{\log \frac{t}{2\pi} - \sum_{p,k} p^{k(\sigma-1)} (\log p) \cos(kt \log p)}. \quad (13)$$

The apparent constancy of the spacing decomposes into two separate smallness mechanisms.

The winding varies only logarithmically. The baseline $\Delta t_0 = 2\pi / \log(t/2\pi) = \pi / \theta'(t)$ equals the mean gap between nontrivial zeros. Its fractional drift per pony is $2\pi / (tL^2)$: about 4.5% at $t = 40$ and 0.26% at $t = 200$. Halving the spacing requires $t \mapsto t^2/2\pi$ (from 40 to ≈ 255); within any window short compared with t the eye sees a constant.

The undulation is exponentially suppressed on the left, and coherent. The modulation of the denominator obeys $|U'| \leq \sum_{p,k} p^{k(\sigma-1)} \log p = |\zeta'(1-\sigma)|$: numerically 0.19813 at $\sigma = -2$ (about 11% of L), 0.02857 at $\sigma = -4$, 0.00603 at $\sigma = -6$. Moreover — this is the spectral-gap statement of [32] in ponyo language — the slowest prime mode has frequency $\log 2$, hence wavelength $2\pi / \log 2 = 9.0647$, which exceeds the ponyo spacing $2\pi/L$ whenever $L > \log 2$, i.e. for all $t > 4\pi$. Adjacent ponys are therefore pushed largely in unison: the undulation *bends* the ponys but barely *compresses* them.

Table 1 compares measured gaps of the cyan ponys at $\sigma = -2$ with formula (13). The bare winding already predicts within a few percent; adding the undulation correction (evaluated at the midpoint) brings most gaps within 1%, the remainder being the midpoint-versus-average approximation. At $t \approx 200$ the measured gaps were 1.76–1.88 against $2\pi/L = 1.81$; at $\sigma = -5$ the bare formula was already accurate to $< 1.5\%$.

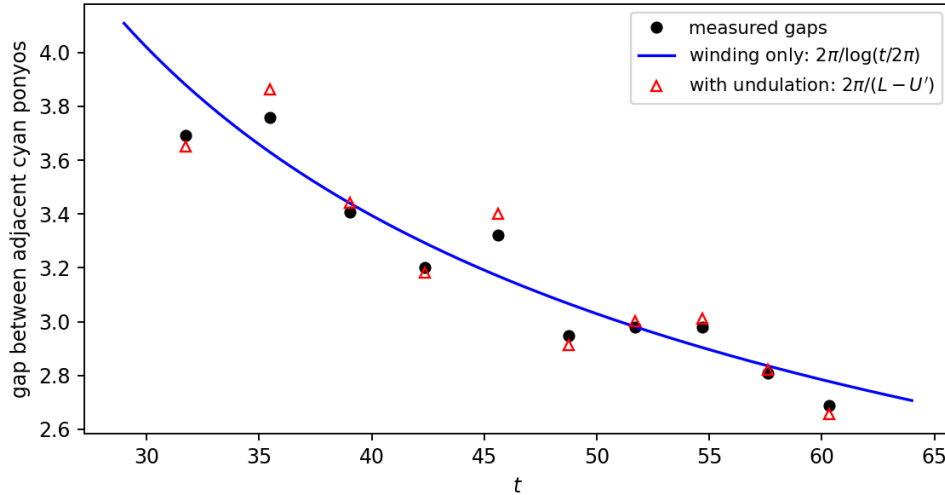


Figure 5: Measured cyan-ponyo gaps at $\sigma = -2$ against the winding-only law $2\pi / \log(t/2\pi)$ and the undulation-corrected formula (13).

Table 1: Measured cyan-ponyo gaps at $\sigma = -2$ versus formula (13).

midpoint t	gap (meas.)	$2\pi/L$	err.	$2\pi/(L - U')$	err.
31.75	3.691	3.879	+5.1%	3.682	-0.3%
39.05	3.407	3.439	+1.0%	3.433	+0.8%
42.36	3.202	3.293	+2.8%	3.186	-0.5%
48.75	2.949	3.067	+4.0%	2.927	-0.8%
60.34	2.688	2.778	+3.3%	2.663	-0.9%

The parabolic tilt of (8) was verified separately by tracing one ponyo leftward: at $\sigma = -8$ the predicted upward displacement is +0.473 against a measured +0.465; at $\sigma = -4$, 0.132 against 0.131.

4.2 One ponyo, one zero

Tracing the cyan ponyo anchored at $t_c = 40.7000$ (where $\theta(t_c) = (5 + \frac{1}{2})\pi$, a *Gram midpoint*) from $\sigma = -8$ into the strip gives

σ	-8	-4	-2	0	0.45	0.49
$t(\sigma)$	41.165	40.834	40.757	40.835	40.909	40.9167

heading precisely into the seventh zero $\gamma_7 = 40.918719\dots$: the ponyo grounds at “its” zero. Counting is forced: the number of cyan ponyos below height T is $\theta(T)/\pi + O(1) \approx N(T)$ by (2), one ponyo per zero, and the cyan anchors $\theta = (m + \frac{1}{2})\pi$ interlace the Gram points $\theta = m\pi$ [4]; Gram’s law (“one zero per Gram interval”) is the statement that ponyo and zero stay matched, and its failures [6] are exactly the ponyo–zero mismatches quantified next. (The same θ /Gram structure underlies Turing’s method for certifying the completeness of zero lists [12].)

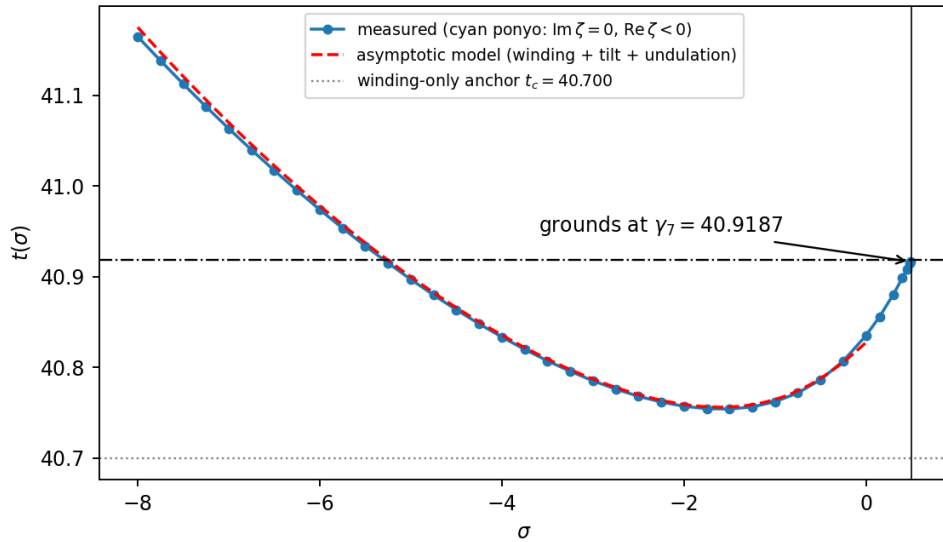


Figure 6: One ponyo traced from $\sigma = -8$ into the strip: the parabolic leftward tilt, the prime undulation, and the grounding at $\gamma_7 = 40.918719$.

4.3 Orientation of the grounding

Near a simple zero, $\zeta(s) \approx \zeta'(\rho)(s - \rho)$: the local color wheel is rotated by $\arg \zeta'(\rho)$, and crossing horizontally just above the zero shows the pair $[\arg \zeta'(\rho) + \pi] \rightarrow [\arg \zeta'(\rho)]$. The observed cyan-to-red pattern is thus the statement $\arg \zeta'(\rho) \approx 0$ — a condition on the derivative, not on $\beta = \operatorname{Re} \rho$; a zero of any function at $\sigma = 0.7$ with positive-real derivative grounds cyan-to-red at 0.7. Restricting the colors therefore cannot convert grounding into a constraint on zero locations.

Why, then, is the pattern so uniform for ζ ? On the line, $\zeta(\frac{1}{2} + it) = e^{-i\theta(t)} Z(t)$ gives the exact orientation law

$$\zeta'(\rho_n) = -i e^{-i\theta(\gamma_n)} Z'(\gamma_n). \quad (14)$$

Since θ advances by $\approx \pi$ per zero while Z' alternates in sign, the rotations nearly cancel and $\arg \zeta'(\rho_n)$ is typically locked near 0° — the same regularity as Gram's law, with deviation $\pi \times (\text{the counting fluctuation})$. Measured: $+9.1^\circ, -12.6^\circ, +19.2^\circ, -30.8^\circ, +32.9^\circ, -16.8^\circ$ for $n = 1, \dots, 6$, drifting; near the first Gram-law violations, $n = 127$ ($\gamma \approx 282.47$) reaches -91.1° and $n = 136$ ($\gamma \approx 295.57$) reaches $+91.2^\circ$: past 90° , where the fine-scale color pair is no longer cyan–red. Since S is unbounded, the pattern is a typical rule, not a theorem. For ξ the same computation gives $\arg \xi'(\rho_n) = \pm 90.0000^\circ$ exactly and alternately (reality of Ξ on the line with Ξ' alternating), reproducing the interlocutor's alternating green–blue observation; the constancy for ζ and the alternation for ξ are two faces of the presence or absence of the winding factor $e^{-i\theta}$.

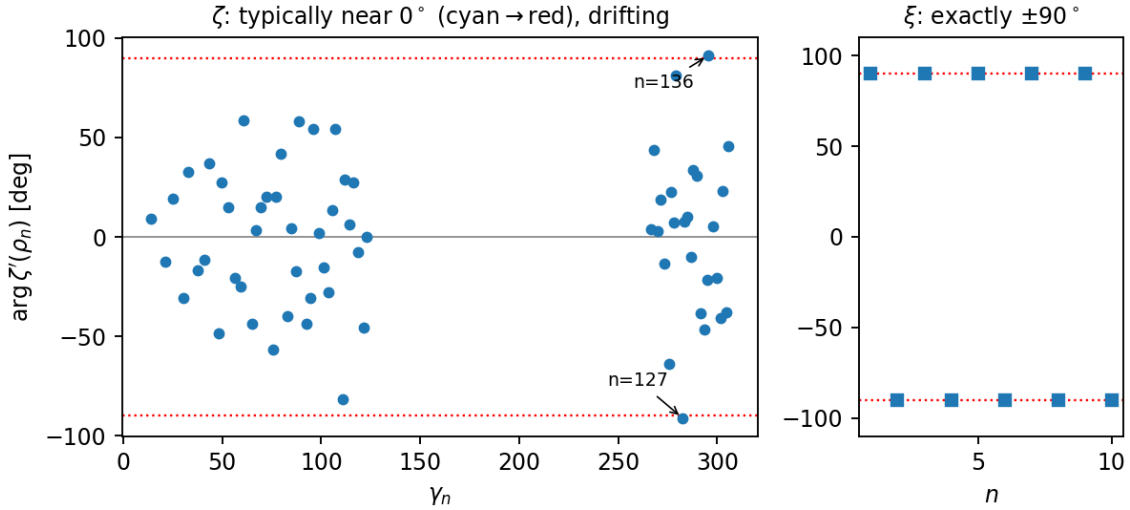


Figure 7: Orientation of the grounding. Left: $\arg \zeta'(\rho_n)$ drifts about 0° and crosses $\pm 90^\circ$ near the first Gram-law violations ($n = 127, 136$). Right: for ξ the alternation $\pm 90^\circ$ is exact.

5 The two terms are the two sides of the explicit formula

5.1 The winding is archimedean

The winding rate is not a generic complex-analytic phenomenon but the phase of the archimedean Euler factor: $\arg[\pi^{-s/2}\Gamma(s/2)]$ on the line is $\theta(t)$, by definition, with $\theta'(t) \approx \frac{1}{2} \log(t/2\pi)$ from Stirling. The finite-prime factors contribute only bounded phase oscillation ($\arg \zeta$ is bounded for

$\sigma \geq 1 + \delta$); the factor χ itself arises from Poisson summation on $\mathbb{R}$. By the argument principle, the winding difference between the two edges of the strip counts the zeros — equation (2) — so the smooth density of zeros (Weyl term, in the spectral reading of [25]) is archimedean, and the fluctuation S is where the primes act. Dyson proposed to read this set-to-set Fourier duality of zeros and prime powers as a one-dimensional quasicrystal [27].

5.2 The Weil formula in winding form, verified

For even h , analytic and decaying on $|\operatorname{Im} r| \leq \frac{1}{2} + \varepsilon$, with $g(u) = \frac{1}{2\pi} \int h(r) e^{-iur} dr$, Weil's explicit formula [11] can be arranged (the $\log \pi$ terms cancelling against the digamma) as

$$\sum_{\rho} h(\gamma_{\rho}) = h\left(\frac{i}{2}\right) + h\left(-\frac{i}{2}\right) + \frac{1}{\pi} \int_{-\infty}^{\infty} h(r) \theta'(r) dr - 2 \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} g(\log n) : \quad (15)$$

zero side equals pole terms, plus the *winding density*, plus the *prime waves*. With a Gaussian window of width 1 at $T = 30$ the three pieces were computed independently:

$$\underbrace{1.85433523}_{\text{zeros (40 used)}} = \underbrace{1.24686806}_{\text{winding}} + \underbrace{0.60746717}_{\text{primes}}, \quad \text{discrepancy } 2.6 \times 10^{-14},$$

the pole terms being $O(e^{-450})$. The prime waves account exactly for the excess of the window over the smooth density caused by the zero $\gamma_4 = 30.425$ sitting inside it. Conversely, formula (3) evaluated with the first 100 zeros reconstructs the prime staircase: at $x = 28.6/29.4$ the true values $25.109/28.477$ are matched by $24.795/28.809$ (the jump $\log 29 = 3.3673$ emerging from interference), and similarly across the jumps $\log 47$, $\log 53$. On the counting side,

t	10^3	10^4	10^5
$N(t)$	649	10142	138069
$\theta/\pi + 1$	648.616	10142.965	138068.558
$S(t)$	+0.384	-0.965	+0.442

against Selberg's RMS $\sqrt{\log \log t}/(\pi\sqrt{2}) \approx 0.31\text{--}0.35$: the winding carries essentially the entire count; the primes supply a jitter of less than one zero.

6 Why the critical line is critical

Write $\log \zeta(\sigma + it) = \sum_{p,k} k^{-1} p^{-k\sigma} e^{-ikt \log p}$. Unique factorization makes the frequencies $\{k \log p\}$ linearly independent over $\mathbb{Q}$, so the phases behave like independent uniform variables (Kronecker–Weyl) and variances add:

$$\operatorname{Var}[\log |\zeta(\sigma + it)|] \approx \frac{1}{2} \sum_p p^{-2\sigma} \quad (+\text{higher prime powers}), \quad (16)$$

convergent precisely for $\sigma > 1/2$. Empirically, over $t \in [400, 1200]$: measured variances 0.039, 0.233, 0.735 at $\sigma = 2, 1, 0.6$ against theoretical 0.039, 0.237, 0.744; at $\sigma = 0.5$ the measured 1.434 already exceeds the cutoff prediction 1.165, the excursions toward the zeros fattening the lower tail. On the line the cutoff sum is Mertens' theorem [2]: $\sum_{p \leq X} p^{-1} = \log \log X + M + o(1)$, verified at $X = 10^6$ as 2.887328 against 2.887289 (difference 3.9×10^{-5} ; the theorem carries an $o(1)$ term, so this closeness is a finite- X consistency, not an identity) — the slowest possible divergence.

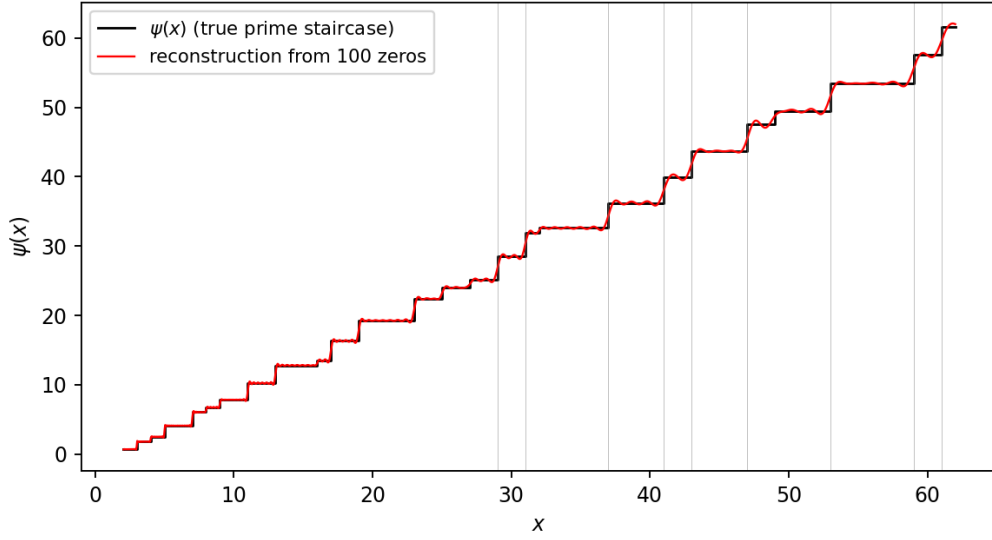


Figure 8: The explicit formula (3) evaluated with the first 100 zeros: waves of frequencies $\{\gamma_n\}$ interfere to build the jumps of the prime staircase.

ζ is thus protected against vanishing by two independent mechanisms. To the right of the line the variance (16) is finite: the fluctuation saturates to a limiting distribution (Bohr–Jessen [7]); its tails are heavy enough for the values to be dense (and universal: Voronin [16]) in $1/2 < \sigma < 1$, but reaching $-\infty$ — a zero — is precisely what RH forbids, equivalently the convergence of $\sum \Lambda(n)n^{-s}$ for $\sigma > 1/2$. To the left of the line the functional equation (1) adds the deterministic drift $(\frac{1}{2}-\sigma)\log(t/2\pi)$ to $\log|\zeta|$; measured at $\sigma = 0.2$, mean 1.540 against predicted 1.521. The drift coefficient vanishes at $\sigma = 1/2$ ($|\chi| = 1$) and the variance diverges at $\sigma = 1/2$: *both protections fail at the same abscissa*. They are, however, not independent: by the functional equation the zeros with $\sigma < 1/2$ are paired exactly with those with $\sigma > 1/2$, so the left protection is the mirror image of the right one supplemented by the deterministic factor $|\chi|$; and both halves of the coincidence — the mirror center and the variance boundary $2\sigma = 1$ — trace to the same pole at $s = 1$ that governs the density of primes. What the computation exhibits is one criticality seen from two sides, not two accidents. Nor does divergence of the variance by itself produce zeros: that infinitely many zeros actually lie on the line is Hardy’s theorem [5]; the variance picture is consistent with it, no more.

7 The ponyo tips as a symmetry-breaking transition

7.1 The de Bruijn–Newman frame

The relevant symmetry is the $\mathbb{Z}_2$ mirror $s \mapsto 1 - \bar{s}$ of the functional equation, whose fixed axis is the critical line; equivalently, the reality of $\Xi(t) = \xi(\frac{1}{2} + it)$, whose roots are symmetric under conjugation. The zero *set* is always symmetric (zeros come in mirror pairs); RH is the statement that every zero is individually a fixed point — the symmetric phase — with order parameter $\delta = \beta - \frac{1}{2}$. The control parameter is the heat deformation H_λ of Ξ (de Bruijn [10], Newman [17]):

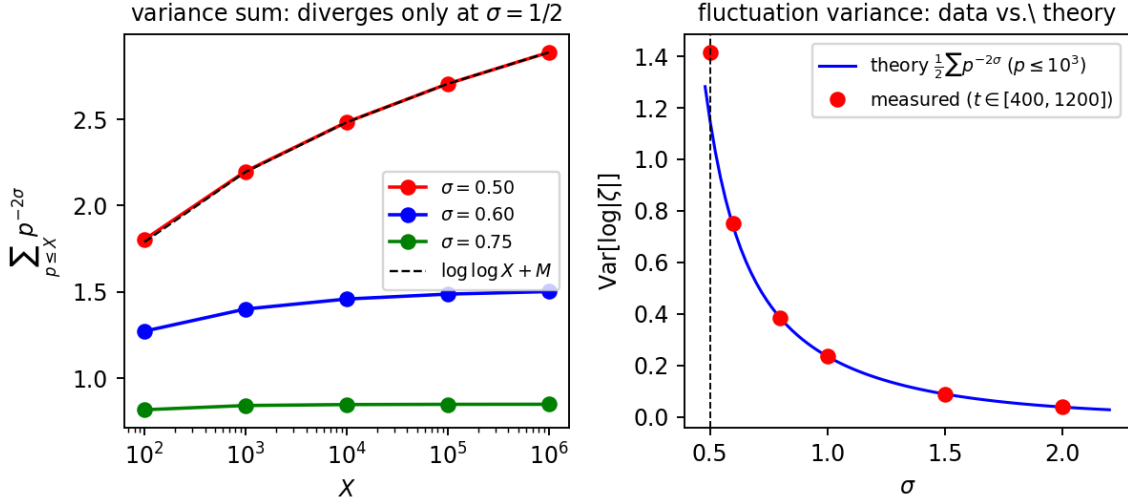


Figure 9: Criticality of the line. Left: the prime-noise variance sum $\sum_{p \leq X} p^{-2\sigma}$ diverges only at $\sigma = 1/2$, following Mertens' $\log \log X + M$. Right: measured $\text{Var}[\log |\zeta(\sigma + it)|]$ against $\frac{1}{2} \sum_p p^{-2\sigma}$.

for $\lambda \geq \Lambda$ all roots are real, below Λ conjugate pairs appear. The theorems

$$\begin{aligned} \text{RH} &\iff \Lambda \leq 0 \quad (\text{de Bruijn, Newman}), \\ \Lambda &\geq 0 \quad (\text{Rodgers-Tao [30]}), \quad \Lambda \leq 0.22 \quad (\text{Polymath15 [29]}) \end{aligned}$$

combine to: *RH holds if and only if $\Lambda = 0$* — the zeta function sits exactly on the phase boundary (“the Riemann Hypothesis, if true, is only barely so” [17]). Numerically, RH is verified to height $3 \cdot 10^{12}$ [31].

7.2 A finite- N pitchfork on the actual zeros

Under the flow the real roots obey $\dot{x}_j = \sum_{k \neq j} 2/(x_j - x_k)$; running λ downward on the 120 points $\{\pm\gamma_1, \dots, \pm\gamma_{60}\}$, the closest pair, #34–#35 ($\gamma = 111.030, 111.875$; gap 0.845), collides at $\lambda_c \approx -0.096$ and splits into a conjugate pair. The measured local laws,

$$r^2 = 7.57 (\lambda - \lambda_c) \quad [\text{isolated-pair theory 8}], \quad b^2 = 2.06 (\lambda_c - \lambda) \quad [\text{theory 2}],$$

with fitted exponents 0.52 ± 0.02 and 0.49 ± 0.01 (uncertainties estimated by varying the fit window; no full systematic-error analysis was performed) and coinciding critical points (-0.0963 vs. -0.0959), realize the mean-field pitchfork; the order parameter tracks $b = \sqrt{2(\lambda_c - \lambda)}$ (e.g. 0.2026 vs. 0.2002). In the s -plane the split pair is the mirror doublet $\sigma = \frac{1}{2} \mp 0.361$ at $t \approx 111.4$: the set symmetric, the members not — textbook spontaneous $\mathbb{Z}_2$ breaking, with a second pair beginning to split as λ decreases further. *Caveat*: this truncated flow illustrates the local mechanism only; it computes nothing about Λ itself, for which the rigorous anchors are the theorems above. In particular the identity of the colliding pair (#34–#35) and the value $\lambda_c \approx -0.096$ are themselves truncation artifacts; only the local square-root form is meaningful. Consistently with sitting at a transition, the unfolded nearest-neighbor spacings of the first 60 zeros have mean 0.986 and variance 0.110 — indicating strong level repulsion, far from Poisson (1) and distinct from a lattice

(0); with only 59 gaps at low height, the difference from the GUE value 0.178 [14, 19] lies within roughly two standard errors plus finite-height corrections, so no precise test of GUE is claimed. Lehmer pairs [13] are near-collisions, the moments when the symmetry creaks (cf. [22]).

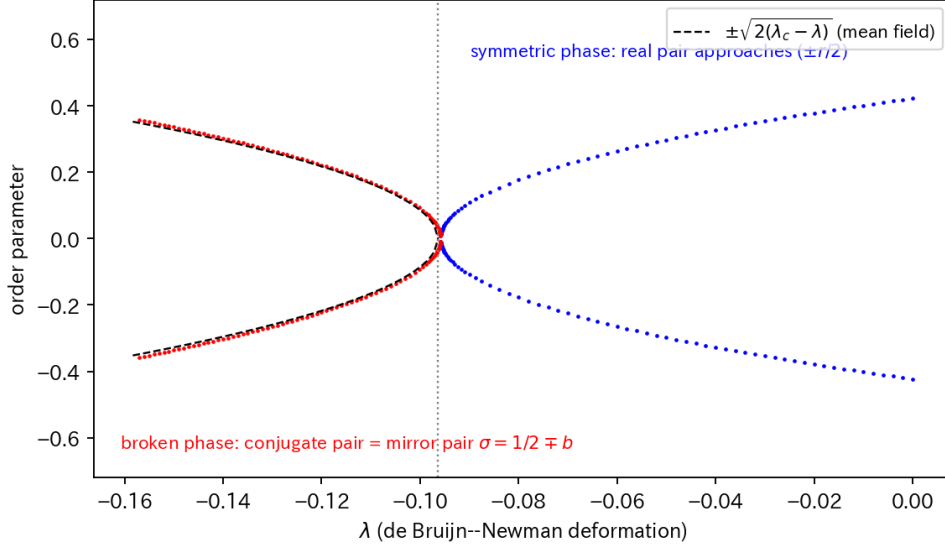


Figure 10: The finite- N de Bruijn–Newman flow on the actual first 60 zeros: the closest pair collides at $\lambda_c \approx -0.096$ and splits into the mirror doublet $\sigma = \frac{1}{2} \mp b$, following the mean-field pitchfork $b = \sqrt{2(\lambda_c - \lambda)}$.

7.3 The trivial zeros: the frozen symmetric lattice

The trivial zeros are of a different provenance: they are the reflections of the poles of $\Gamma(s/2)$ — purely archimedean, prime-free, absent from ξ altogether. Their lattice is exact (spacing 2, count $T/2$, fluctuation zero), their ponyo trajectories carry a prime content of only 2.5% of the phase (measured at $s = -2 + 0.5i$), and their alternating color pairs are the orderly stripes of an unbroken phase. In the flow picture they sit, from the start, in the equally-spaced configuration toward which increasing λ drives all roots: the ground state of the symmetric phase. Their count grows ($T/2$), but they create nothing on the prime side: in the budget of (3) the pole contributes x , the nontrivial zeros an oscillation of envelope $\sqrt{2x} \sum |\rho|^{-2}$ (measured $0.63 \rightarrow 20.0$ for $x = 10 \rightarrow 10^4$), and the trivial zeros $\frac{1}{2} \log(1 - x^{-2})^{-1}$: 5.0×10^{-3} at $x = 10$, 5.0×10^{-9} at $x = 10^4$ — an evanescent vacuum correction. Where a genuine “growth of matter” does attach rigorously is the pole: reading $\zeta(\beta)$ as the partition function of the primon gas [20, 21], the energy $U(\beta) = -\zeta'/\zeta(\beta)$ diverges as $1/(\beta - 1)$ (measured 19.43 vs. 20 at $\beta = 1.05$; 99.42 vs. 100 at 1.01) — a Hagedorn point; and the Bost–Connes system [23], with partition function $\zeta(\beta)$, has a genuine symmetry-breaking phase transition at $\beta = 1$. Growth and its transition live at the pole; criticality of the mirror lives on the line; the trivial zeros are the frozen background.

8 Spectroscopy of the ponyos: an implementation check and a visualization

The companion note [32] isolates, on the critical line, the *fundamental tone* $W(t) = 2^{-1/2-it}$: the unique prime wave of amplitude exactly $2^{-1/2}$, the lowest frequency $\log 2$ of the whole prime spectrum, and the gap $(0, \log 2)$ below it. Equation (7) predicts that these are not merely algebraic facts about a series but *measurable properties of the ponyos*: the displacement field of the ponyos (measured on their centreline comb curves) at abscissa σ should be the superposition (6), with the mode $(p, k) = (2, 1)$ dominant at amplitude $2^{\sigma-1}$ for every $\sigma < 1$, and with no spectral line below $\log 2$. This section reports the measurement — with its status stated first. Equation (4) is an identity: if the code is correct, the measurement below *cannot fail*. It is therefore a check of the implementation and a visualization of the prime comb, not a test of nature; the derivation-testing content of this note lives instead in the $1/t$ residuals of (5), the spacing law (13), the tilt of (8), and the 90° crossings around (14) (Section 4.3).

Method. The phase $\arg \zeta(-2+it)$ was sampled on $t \in [30, 130]$ (step 0.05), unwrapped, and the smooth part $-2\theta(t) + 3.125/t$ of (7) subtracted, together with a fitted slow trend $a + b/t + c/t^2$; the residual is the ponyo-displacement field in phase units. The prime comb up to $n = 13$ was then fitted simultaneously by least squares (a simultaneous fit is essential: a naive single-frequency scan shows a floor of 0.0266 inside the gap, which is exactly the rectangular-window sidelobe of the fundamental, $0.217 \times 0.1256 = 0.0273$).

Result. Table 2 shows the nine fitted lines against the amplitudes $k^{-1}p^{k(\sigma-1)}$ at $\sigma = -2$. After removal of the comb, the scan was extended from v1's $\omega \geq 0.30$ down to $\omega = 0.10$ (the window length 100 limits the resolution to $2\pi/100 \approx 0.063$): at $\sigma = -2$ the residual is 7.5×10^{-7} on $[0.10, 0.30)$ and $\leq 2.3 \times 10^{-5}$ on all of $[0.10, 2.72]$ — in particular the gap $(0, \log 2)$ is empty down to the resolution limit, as [32] requires. The same measurement was repeated at $\sigma = -1$ and $\sigma = -4$ (v1 tested a single abscissa): the fitted fundamental amplitudes are 0.25000/0.12500/0.03125 at $\sigma = -1, -2, -4$, i.e. the law $2^{\sigma-1}$ across abscissae; the residual maxima are 3.9×10^{-4} (at $\sigma = -1$, from higher comb lines outside the fitted set), 2.3×10^{-5} , and 8.2×10^{-6} .

Table 2: Ponyo spectroscopy at $\sigma = -2$, $t \in [30, 130]$. Theory: amplitude $k^{-1}p^{k(\sigma-1)}$.

$n = p^k$	frequency $k \log p$	fitted amplitude	theory
2	0.69315	0.12500	0.12500
3	1.09861	0.03704	0.03704
$4 = 2^2$	1.38629	0.00781	0.00781
5	1.60944	0.00800	0.00800
7	1.94591	0.00292	0.00292
$8 = 2^3$	2.07944	0.00065	0.00065
$9 = 3^2$	2.19722	0.00069	0.00069
11	2.39790	0.00076	0.00075
13	2.56495	0.00045	0.00046
residual after removing the comb: $\leq 1 \times 10^{-5}$ on $[0.30, 2.40]$; gap $(0, \log 2)$ empty			

Three remarks. (i) The line at $n = 4$ *visualizes* the distinction between normalizations: the fitted $0.00781 = \frac{1}{2} \cdot 4^{\sigma-1}$ agrees with the Π -type weights $k^{-1}p^{k(\sigma-1)}$ of (6), not with the naive Dirichlet weight $4^{\sigma-1} = 0.0156$. These weights are, however, fixed a priori by the expansion

of $\arg \zeta$; what the measurement settles is that the implementation reads the correct series — an enactment, not an adjudication, of Remark 3.6 of [32]. (ii) The fundamental tone is, as predicted, the strongest line at every abscissa, with amplitude $2^{\sigma-1}$ ($= 2^{-1/2}$ on the line); its period 9.0647 is the dominant undulation wavelength observed in Section 4.1, and its share of the ponyo-spacing modulation bound at $\sigma = -2$ is $\log 2 \cdot 2^{\sigma-1}/|\zeta'(3)| = 43.7\%$, while on the line it carries $\frac{1}{2}/\sum_{p \leq 10^6} p^{-1} = 17.3\%$ of the critical variance. (iii) To the question of [32] — for which family of waves does a pattern become ζ -like? — the ponyos supply the one known answer: *the phase field of ζ itself* is a pattern whose displacement spectrum is the prime comb. This is a spectroscopy of ζ 's self-portrait, not a construction of the reaction–diffusion system of [32, §7], which remains a blueprint.

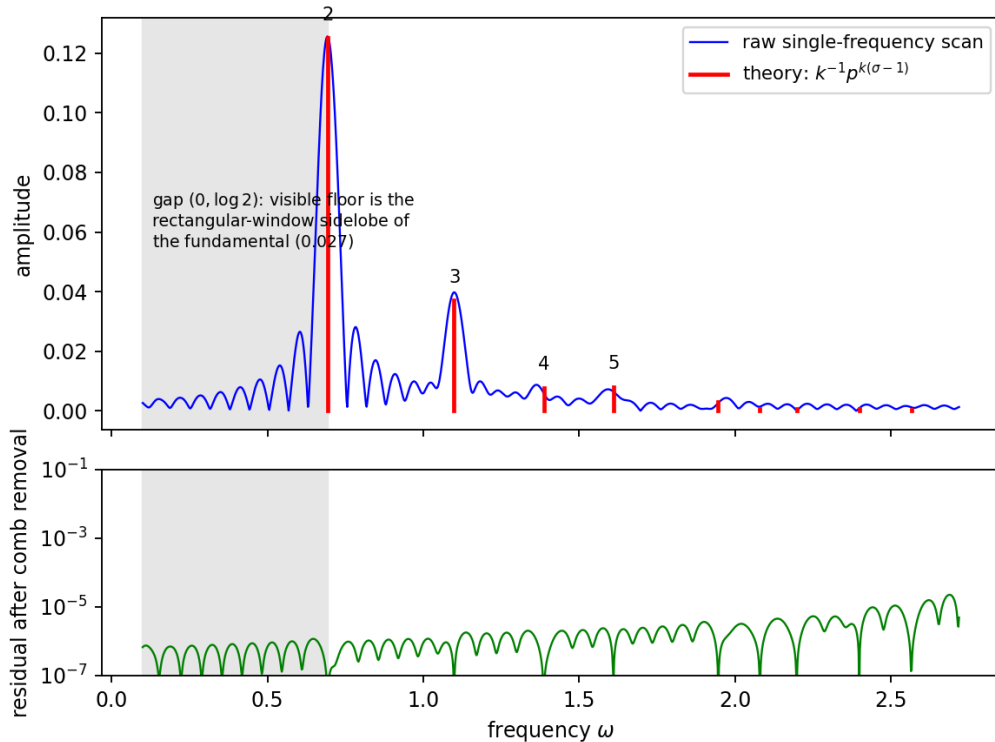


Figure 11: Ponyo spectroscopy at $\sigma = -2$. Top: raw scan of the displacement field with the theoretical prime comb $k^{-1}p^{k(\sigma-1)}$ (red); the apparent floor in the gap is the window sidelobe of the fundamental. Bottom: after simultaneous removal of the comb the residual is $\lesssim 10^{-5}$ throughout the scanned range $[0.10, 2.72]$; the gap $(0, \log 2)$ is empty down to the resolution limit $2\pi/100 \approx 0.063$.

9 What is established here, and in what sense

Two further classical identities were used and are worth displaying. The digamma value

$$\psi_{\Gamma}\left(\frac{1}{2}\right) = -\gamma_E - 2 \log 2 = -1.96351002602142 \dots, \quad (17)$$

a consequence of the Legendre duplication formula, ties the constant of the archimedean background to twice the fundamental frequency at the very center of the line — the winding–undulation

Table 3: Status of the claims of this note.

Claim	Status
Exact comb equation (4)	classical identity; verified to 10^{-20}
Asymptotic comb equation (7), tilt term	elementary (Stirling); verified
Spacing law (13); coherence for $t > 4\pi$	elementary; verified to $\sim 1\%$
One ponyo–one zero; anchors at Gram midpoints	classical counting (2); verified
Orientation law (14); cyan–red typicality	exact identity; “always” is <i>false</i> ($n = 127$)
$\arg \xi'(\rho_n) = \pm 90^\circ$ alternating	exact (reality of Ξ); verified
Vertical grounding local; $\Gamma \subseteq L$ false for ζ	counterexamples computed (§3.4)
Weil formula in winding form (15)	classical (Weil); verified to 10^{-14}
Criticality: two protections vanish at $\sigma = 1/2$	Selberg/Mertens theorems + heuristic argument; matches data (the protections are not fully independent: Section 6)
RH $\Leftrightarrow \Lambda = 0$	theorems [10, 17, 30]
Finite- N pitchfork, exponents $\approx 1/2$	illustration of the local mechanism; <i>not</i> a computation of Λ
Ponyo spectroscopy (three abscissae; gap empty to resolution limit)	implementation check and visualization; not falsifiable given (4)
Infinitely many zeros on the line	theorem (Hardy [5])
Cosmological or material readings	analogy only; the budget of §7.3 constrains the mapping (growth $\leftrightarrow$ pole)
Why the zeros stand on the line	open

pairing of this note, present already at the level of constants (it is identity (6) of [32]). And the mirror-pair invariant $|2^{-s}||2^{-(1-s)}| = \frac{1}{2}$ for all s , balanced ($= 2^{-1/2}$ each) exactly on the line, closes logically but, as §6 of [32] insists, is a reparametrization of $\sigma = \frac{1}{2}$ carrying no arithmetic content. This note claims no new mathematical results; its contribution is the organization, the explicit elementary derivations, and the numerical experiments.

Limitations and reproducibility

(a) The spectroscopy (Section 8) is an implementation check of an identity, not a falsifiable experiment. (b) The gap verification was limited to $\omega \geq 0.30$ in v1; this version extends it to $\omega = 0.10$, but the window length 100 makes frequencies below $2\pi/100 \approx 0.063$ unverifiable in principle. (c) The σ -dependence $2^{\sigma-1}$ of the amplitudes was tested at three abscissae only. (d) The pitchfork exponent uncertainties come from varying the fit window; no full systematic-error analysis was performed, and both the identity of the colliding pair and the value of λ_c are truncation artifacts. (e) Tabulated values were computed in arbitrary precision (`mpmath`, 8–25 digits), the figures separately in double precision (`mpmath.fp`). (f) All computations are short scripts written during the dialogue, re-implementable from the descriptions in the text, but no code repository accompanies this memorandum. (g) Bibliographic details were written from memory and not checked against the sources; no systematic literature search was performed — in particular the level-curve analysis very likely overlaps with Arias de Reyna’s X-ray [26] — so no novelty is claimed even for the numerical experiments. (h) References [32, 33] are non-archival (dialogue documents and temporary URLs), and the compiler of this note is an AI; any external publication must follow the policies of the venue.

10 Concluding remarks

What the interlocutor's twenty years of phase portraits had recorded turns out to be readable, item by item, in a two-term ledger. The ponyos are the level lines of a phase whose smooth rotation is archimedean and whose tremor is the sum of the prime waves; the ledger is the explicit formula; the ponyos ground one-per-zero on the line where the deterministic and stochastic protections of ζ fail together; the tips sit — by theorem, if RH holds — exactly on the boundary $\Lambda = 0$ where the mirror symmetry would break and never does; the trivial zeros are the frozen lattice of the archimedean vacuum; and the ponyos audibly ring with the fundamental tone W , whose amplitude, gap, and overtone weights the spectroscopy of Section 8 measures to four–five digits.

None of this explains why the zeros stand on the line: the reformulations polish the constant $\frac{1}{2}$ from every side — balance of amplitudes, vanishing drift, critical variance, boundary of the heat flow — without answering it, and the honest boundary drawn in [32] stands here unchanged. But one thing has been checked that was only asserted before: the pattern and the primes are not joined by metaphor. The ponyos are a pattern, their spectrum is the prime comb, and the fundamental tone sounds in them, at exactly the amplitude the arithmetic prescribes.

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