The Twist of the Complex Plane

—Two questions about the pole $s = 1$ and the zeros on the critical line—

Toshiki Takahashi*
TML independent researcher

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Abstract

The sister paper [18] calls the coloured bands seen in the phase portrait of $\arg\zeta$ "ponyos" and shows that their phase is the sum of two parts: a smooth *winding* supplied by the archimedean factor, and a prime *undulation* supplied by the Euler product. Building on this decomposition, this note treats the author's two questions as a single computation.

Part I: If the pole $s = 1$ did not exist, would the right side of the critical line return to the left? The answer is: "the right side is already returned to the left, and the pole is the twin of the returning device." (1) The mirror $\{1, 3, 5, \dots\}$ of the left lattice $\{0, -2, -4, \dots\}$ of the poles of $\Gamma(s/2)$ (whose sites $-2, -4, \dots$ are the trivial zeros, while $\zeta(0) = -\frac{1}{2} \neq 0$) is not empty. The simple poles of χ cancel exactly against the mirror zeros, levelling those sites to winding 0; only $s = 1$, whose partner is $\zeta(0) = -\frac{1}{2} \neq 0$, has no cancelling partner and survives as the pole (winding -1; verified numerically). (2) In Z_2, obtained by removing the pole and symmetrising the Γ-factor, a two-sided lattice stands and the pole is replaced by the first right trivial zero ($Z_2'(1) = -1/(4\sqrt{\pi}) = -0.1410473959$, theory = measured; mirror residual 2.3×10^{-19}). (3) Comparing the displacement field of the left ponyo point by point with the prime field at the mirror gives correlation 1.000000, a difference (up to a branch) of RMS 2.7×10^{-5}, and that difference vanishes as t^{-3} (slope -3.00). The trembling on the left is the mirror of the prime field of the right half-plane. (4) The total intensity of the returned wave is exactly equivalent to the pole, with constant $-\gamma_E$. (5) The returning device and the pole are two terms of a single Poisson identity—the modular transform of theta (residual 10^{-27}). (6) For a genuine "no pole"—a Dirichlet $L(s, \chi_5)$—the lattice advances one site to the right and no right lattice stands: the Euler product forbids zeros for $\sigma > 1$.

Part II: If the non-trivial zeros were not lined up on the critical line, would the twist of the complex plane vanish? The answer is: "it does not vanish, because the source of the twist is not the zeros." Measured with a model $F = \zeta \cdot G$ in which a single pair of zeros is moved off the line (preserving the functional equation, real-axis symmetry, and zero

*This note was assembled into the form of a single paper by Claude (Anthropic) on 19 August 2026, from a dialogue between the author and Claude the same day. Both questions are the author's. Part I reworks the same-day note *The Formation of Prime Waves by the Pole-Quake on the Complex Plane* [19]; Part II is a new computation from the second half of the same dialogue. This is a sequel to the sister papers *The Ponyos of the Riemann Zeta Function* [18] and *The Fundamental Tone* $2^{-1/2-it}$ *on the Critical Line* [17], and the terminology and notation—ponyo, grounding, winding, undulation, wedge—follow [18]. This note claims no new mathematical result. Part I is entirely an organisation and numerical check of classical facts (the functional equation, the Poisson summation formula, the Laurent expansion, elementary properties of Dirichlet L-functions); Part II is elementary computation and numerical verification for an explicit model function. The title word "twist" and the phrases "wave return", "anti-grounding", "inverted grounding", etc., are explanatory metaphors; the mathematical claims are confined to the formulae and figures of each section. Euler's constant is written γ_E to avoid confusion with the ordinates γ_n of the zeros. All numerics were computed in Python (mpmath, working precision 8–30 digits; numpy) during the dialogue, and recomputed from scratch for this note. Bibliographic details are given from memory and have not been checked against the originals. References [18, 17, 19] are non-archival, and the editor of this note is an AI. All errors are the editor's. *Corrected version, 2 September 2026:* several errors and internal contradictions of the 19 August version have been corrected; the changes are listed in §21.

count exactly): (7) the functional equation identically forbids the defect's dipole moment, so the far field decays as a quadrupole t^{-3} (measured coefficient -28.69 vs. theory -28.66). (8) What breaks is the ponyos' wiring: one ponyo crosses the critical line at $t = 38.999$ and grounds on the right zero inside the right half-plane; the other overshoots the left zero, turns back at $\sigma^* = 0.3725$—still to the left of the critical line—and grounds on the left zero from the right. (9) The line loses its grounding and $|Z_F|$ stops at a floor of 0.0586—the completed form of a would-be Lehmer pair. (10) The exact $\pm 90°$ alternation is replaced by the supplementary law $\arg \xi'(\rho_L) + \arg \xi'(\rho_R) = 180°$, of which the former is the degeneration on the line. (11) The pole constant $-\gamma_E$ moves by only 7.3×10^{-6}. (12) The price is paid in arithmetic: the error term of $\psi(x)$ swings at $x^{0.8}$.

Putting the two answers together yields one picture. The twist is made by the pole and the archimedean factor, returned by the mirror, and received by the zeros. Remove the pole and a lattice stands on the right but the tie to the primes is cut; move the zeros off the line and the wiring is rearranged and the error of the prime staircase grows. In either case the one who sends the bill is the Euler product.

1 Introduction: two questions

The guiding principle of the sister paper [18] was a two-term decomposition of the phase. Along any vertical line in the complex plane, the phase of ζ is the sum of

- the *winding*—a smooth, deterministic rotation supplied by the archimedean factor $\pi^{-s/2}\Gamma(s/2)$ of the completed zeta, and
- the *undulation*—a superposition of prime waves $k^{-1}p^{k(\sigma-1)} \sin(kt \log p)$ supplied by the Euler product through the functional equation.

The winding sets the number and mean spacing of the ponyos; the undulation bends them. And each ponyo grounds, one per zero, on a non-trivial zero and ends there. (For phase portraits in general, see Wegert [15].)

Granting this picture, two naive questions arise. Both were raised by the author while looking at phase portraits.

> **Question I.** The mirror of the functional equation $\xi(s) = \xi(1-s)$ is $s \leftrightarrow 1-s$. Reflecting the left trivial-zero lattice $\{-2, -4, -6, \dots\}$ gives $\{3, 5, 7, \dots\}$, and the site $s = 0$ (where $\zeta(0) = -\frac{1}{2}$ sits, at the first pole of $\Gamma(s/2)$) is reflected to $s = 1$; yet nothing seems to be there and the right half-plane looks like a uniform red sea. If the pole $s = 1$ did not exist, would ponyos also extend to the right? In other words, **is it precisely because of the pole that the right-side wave is returned to the left?**
>
> **Question II.** The ponyos ground on the critical line and end there. If the non-trivial zeros were not lined up on the critical line—if the real part of the zeros did not settle at 1/2—**would the twist of the complex plane vanish?** Or would it take another form?

The two look separate but probe the two ends of the same structure. The pole $s = 1$ is the source of the twist; the zeros are its receivers. So computing what happens when one is removed should simultaneously reveal the role of the other. With this outlook, Part I puts Question I on a computational footing, Part II does the same for Question II, and Part III sets the two answers side by side.

Everything can be done with the tools of the sister paper alone: the functional equation, the Poisson summation formula, the explicit formula, and the winding number of the phase portrait.

Part I

If the pole $s = 1$ did not exist, would the right side of the critical line return to the left?

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2 The mirror lattice is not empty—it is neutralised

First we look inside the fact that the right half-plane appears as a "uniform red sea". Write the functional equation as

$$\zeta(s) = \chi(s)\,\zeta(1-s), \qquad \chi(s) = \pi^{s-1/2}\,\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}. \tag{1}$$

The numerator $\Gamma\left(\frac{1-s}{2}\right)$ has simple poles at $\frac{1-s}{2} = 0, -1, -2, \ldots$, which is exactly the mirror lattice $s = 1, 3, 5, 7, \ldots$ of the pole lattice $\{0, -2, -4, \ldots\}$ of $\Gamma(s/2)$. So the appearance of ζ at each site is decided by the product of χ's pole with the value of $\zeta(1-s)$ (Table 1).

Table 1: What happens at each site of the mirror lattice. Winding is the measured $\Delta \arg/2\pi$ along a circle of radius 0.3 (400 points).

site s	$\zeta(1-s)$	$\chi(s)$	appearance of $\zeta(s)$	winding (measured)
1	$\zeta(0) = -\frac{1}{2} \neq 0$	simple pole	pole (anti-grounding)	-1.0000
3	$\zeta(-2) = 0$	simple pole	ordinary point $\zeta(3) = 1.2020569$	-0.0000
5	$\zeta(-4) = 0$	simple pole	ordinary point $\zeta(5) = 1.0369278$	-0.0000
7	$\zeta(-6) = 0$	simple pole	ordinary point $\zeta(7) = 1.0083493$	-0.0000

Here is the reading. At $s = 3, 5, 7, \ldots$ the pole of χ cancels exactly against the mirror zero $\zeta(-2) = \zeta(-4) = \cdots = 0$, levelling the site to an ordinary point of winding 0. But at $s = 1$ the partner is $\zeta(0) = -\frac{1}{2} \neq 0$, and no cancellation occurs. The pole of χ comes out into the open—this is the pole of ζ.

In the phase portrait, a wedge of each colour quadrant does gather at $s = 1$; but because the winding is -1 the colour order is reversed, giving the only "anti-grounding" in the plane in the sense of [18], §3.3 (the white star in Figure 1a).

So the first answer to Question I is this. **The right-side ponyos did not "go somewhere". The first site of the mirror lattice is occupied by the pole, and the remaining sites are levelled by cancellation, winding and all, and they are there.**

3 The world with the pole removed—in Z_2 the right lattice stands

3.1 Construction and verification

What if we actually remove the pole? Consider the function obtained by removing the pole and symmetrising the Γ-factor,

$$Z_2(s) := \frac{\xi(s)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)} = \frac{\frac{1}{2}s(s-1)\,\pi^{-s/2}\,\zeta(s)}{\Gamma\left(\frac{1-s}{2}\right)}. \tag{2}$$

Z_2 is entire and satisfies the mirror symmetry $Z_2(s) = Z_2(1-s)$ (residual 2.3×10^{-19} at 6 random points). It shares the non-trivial zeros with ζ and has the *two-sided* trivial lattice

$$\{0, -2, -4, \ldots\} \cup \{1, 3, 5, 7, \ldots\}$$

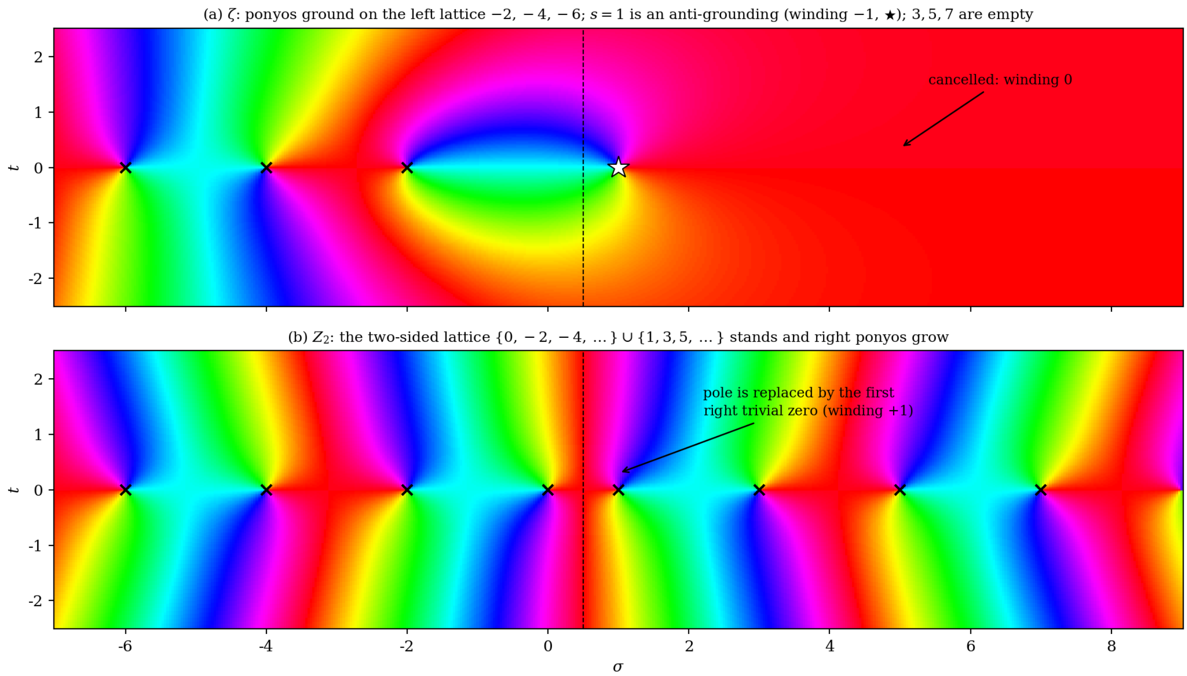

Figure 1: Comparison near the real axis. Hue is the argument (red $=0$, cyan $=\pi$). (a) ζ: ponyos ground on the left lattice $-2,-4,-6$. $s=1$ is an anti-grounding (winding -1, ★), and $3,5,7$ are levelled by cancellation to ordinary points of winding 0. (b) Z_2: the two-sided lattice stands, and the pole is replaced by the first right trivial zero $s=1$ (winding $+1$). $s=0$ revives as a trivial zero. The right ponyo is the mirror of the left, with red and cyan fixed and green and blue swapped.

($0\leftrightarrow 1$, $-2\leftrightarrow 3$, $-4\leftrightarrow 5$, fully symmetric about $\frac{1}{2}$). Numerically $Z_2(0)=Z_2(1)=Z_2(3)=Z_2(5)=Z_2(7)=Z_2(-2)=Z_2(-4)=0$, and the winding is $+1$ at both $s=1$ and $s=3$ (against -1 and 0 for ζ). That is, **the pole is replaced by the first right trivial zero**, and $s=0$, which was eaten by the mirror of the pole in ζ, revives as a trivial zero.

The derivatives also match the theoretical values:

$$Z_2'(1) = -\frac{1}{4\sqrt{\pi}} = -0.1410473959 \quad (\text{theory} = \text{measured, 10 digits}),$$

$$Z_2'(3) = +0.323811, \quad Z_2'(5) = -0.592753, \quad Z_2'(7) = +1.15592.$$

The alternating sign is the mirror image of the alternation of $\zeta'(-2n)$ on the left lattice, and means that on the right sites too the colour pairs swap alternately (Figure 1b).

3.2 How the right ponyo grows

From $Z_2(s)=Z_2(1-s)$ and reality on the real axis, in the upper half-plane

$$\arg Z_2(1-\sigma+it) = -\arg Z_2(\sigma+it). \tag{3}$$

That is, the right half is the mirror of the left, with only the hue inverted: red (0) and cyan (π) are fixed, while green ($\pi/2$) and blue-violet ($3\pi/2$) are swapped. The right ponyo extends from $1,3,5,7,\dots$ to $\sigma\to+\infty$, with the same spacing and thickness at the same height as the left (Figures 1b, 2).

Substituting the reflection formula $1/\Gamma\left(\frac{1-s}{2}\right)=\Gamma\left(\frac{1+s}{2}\right)\sin\left(\frac{\pi(1-s)}{2}\right)/\pi$ gives, for $\sigma>1$, the exact comb equation as the complete mirror of [18] eq. (4):

$$\arg\Gamma\left(\frac{1+s}{2}\right) + \arg\sin\left(\frac{\pi(1-s)}{2}\right) + \arg\big(s(s-1)\big) - \frac{t}{2}\log\pi + \arg\zeta(s) \equiv c \pmod{2\pi}. \tag{4}$$

The sign of $\sin(\pi(1-s)/2)$ flips each time it passes an odd integer $1, 3, 5, \ldots$, producing the alternation of colour pairs at each right site—the exact mirror of the role played by $\sin(\pi s/2)$ at even sites on the left.

The same holds for the undulation. For $\sigma > 1$, $\arg\zeta(s) = -\sum_{p,k} k^{-1}p^{-k\sigma}\sin(kt\log p)$ converges absolutely, and at the mirror point $\sigma \leftrightarrow 1-\sigma$ the amplitude matches exactly as $p^{k(\sigma-1)} \leftrightarrow p^{-k\sigma}$. Hence the right ponyo undulates with exactly the same prime comb, the same fundamental $\log 2$, as the left. The slope term $\sigma(\sigma-1)/2$ is invariant under $\sigma \leftrightarrow 1-\sigma$, so the right ponyo curves up to the upper right in the same parabolic shape.

How a mirror pair meets the critical line. One consequence of the symmetrisation deserves to be stated explicitly, because it changes the picture of grounding. By (3) and reality on the real axis, $Z_2(\frac{1}{2}+it) = \Xi(t)/|\Gamma(\frac{1}{4}+\frac{it}{2})|^2$ is *real*: the critical line itself is alternately a red and a cyan core, with the colour flipping at each zero. At a zero on the line the cyan and red rays are therefore vertical (along the line) and the green and blue rays are horizontal. A cyan core arriving from the left cannot enter a zero from the side; it meets its mirror image, the right cyan core, at a saddle point of $\arg Z_2$ on the line (an extremum of $Z_2(\frac{1}{2}+it)$ between two consecutive zeros where $Z_2 < 0$), and the merged core runs along the line to the two zeros bounding that segment (Figure 2, □). One mirror pair of ponyos thus serves *two* zeros. The count agrees: along $\sigma = -2$ one has $\Delta\arg Z_2(-2+it) \approx -\theta(t) \approx -\pi N(t)$, half of the $-2\theta(t)$ of ζ, so Z_2 has half as many left ponyos as ζ at the same height, the other half arriving from the right.

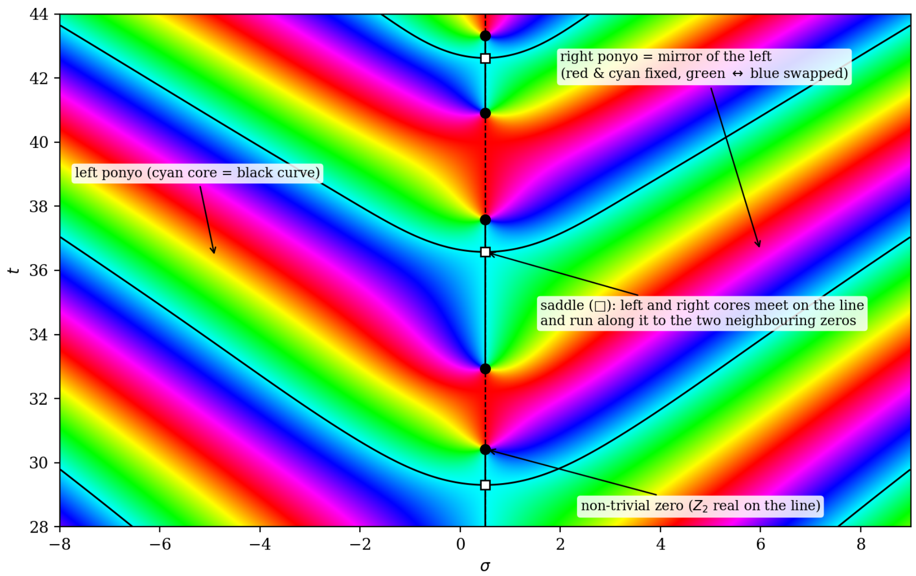

Figure 2: Phase portrait of Z_2 ($t \in [28, 44]$); black curves are the cyan cores $\{\mathrm{Im} = 0,\ \mathrm{Re} < 0\}$. The left and right ponyos form a mirror pair and flow into the critical strip. Since Z_2 is real on the critical line, the line itself is alternately red and cyan between the zeros (black dots): the cyan cores of a mirror pair meet the line at a saddle point (□; $t = 29.31$, 36.58, 42.62) between two consecutive zeros, and the merged core runs along the line to those two zeros, whose green/blue rays are horizontal. The right ponyo is the mirror of the left with green and blue swapped. (Figure redrawn for the corrected version.)

4 The wave return—as an identity, not a metaphor

So far the question was "what is the right-side lattice doing". Now we check the second half of Question I—**is the pole returning the right-side wave to the left?**—as an identity, not a metaphor.

The asymptotic comb equation of [18] says that the undulation of the left ponyo is $U(\sigma, t) = \arg\zeta(1-s)$. That is, the left undulation ought to be the phase field of the right half-plane (where the Euler product converges absolutely—the primes' homeland, so to speak) returned by the functional equation through the mirror $s \mapsto 1-s$. We check this point by point.

For $t \in [30, 130]$ (step 0.05, 2001 points), set

$$L(t) := \big[\operatorname{unwrap}\arg\zeta(-2+it)\big] - \Big[-2\theta(t) + \frac{3.125}{t}\Big], \qquad R(t) := -\arg\zeta(3+it).$$

L is the displacement field of the left ponyo ($\sigma = -2$, winding and slope removed; the term $3.125/t = [\sigma(\sigma-1)/2 + \frac{1}{8}]/t$ at $\sigma = -2$ is the first Stirling correction of $\arg\chi(\sigma+it)$ beyond $-2\theta(t)$), and R is the sign reversal of the prime field at the mirror point $\sigma = 3$. The results (Figure 3):

- correlation coefficient 1.000000; the branch constant is $\operatorname{mean}(L-R)/2\pi = 2.999997$, i.e. exactly $2\pi \times 3$;
- the RMS of the difference after removing $2\pi \times 3$ is 2.72×10^{-5} (1.11×10^{-4} at $t = 30$, 1.36×10^{-6} at $t = 130$);
- the power-law fit of that difference has slope -2.999—the higher Stirling term $O(t^{-3})$ itself; the prime component matches completely.

This re-reads the verification of [18] eq. (7) as a "mirror identity" and is not an independent new fact, but it is the most direct display of the claim that **the trembling of the left ponyo is nothing but the prime field of the right half-plane returned by the functional equation through the mirror**. The amplitude correspondence $p^{k(\sigma-1)} = p^{-k(1-\sigma)}$ is another face of the same identity.

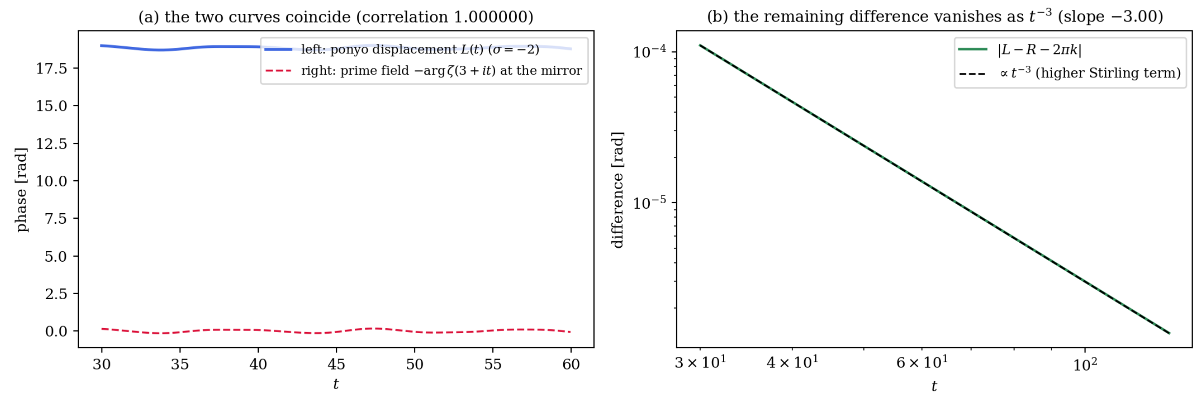

Figure 3: Verification of the wave return. (a) The displacement field $L(t)$ of the left ponyo and the prime field $-\arg\zeta(3+it)$ at the mirror overlap indistinguishably (correlation 1.000000). (b) The difference, up to a branch, vanishes as t^{-3}—all that remains is the higher archimedean term.

5 The pole is the total intensity of the returned wave

We measure the collective strength of the returned wave as $s \to 1^+$. With the von Mangoldt weight, $\sum_n \Lambda(n) n^{-s} = -\zeta'/\zeta(s)$, and from the Laurent expansion $\zeta(s) = \frac{1}{s-1} + \gamma_E + O(s-1)$,

$$-\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1} \longrightarrow -\gamma_E \qquad (s \to 1^+). \tag{5}$$

The measurements:

s	$-\zeta'/\zeta(s)$	$1/(s-1)$	difference
1.5	1.5052354	2	-0.4947646
1.2	4.4583372	5	-0.5416628
1.1	9.4410364	10	-0.5589636
1.05	19.432034	20	-0.5679658
1.02	49.426515	50	-0.5734853
1.01	99.424655	100	-0.5753454

It converges linearly to $-\gamma_E = -0.5772157$, and the measured local slope 0.1865 agrees well with the theoretical value $\gamma_E^2 + 2\gamma_1 = 0.1875462$ ($\gamma_1 = -0.0728158$ is the Stieltjes constant). That is, **the logarithmic divergence of the sum of the prime waves is exactly equivalent to a simple pole of residue exactly 1, and its constant term is written in the closed form $-\gamma_E$, Euler's constant.**

The same happens with the primes-only weight. For the prime zeta $P(s) = \sum_p p^{-s}$,

$$P(s) - \log \frac{1}{s-1} \longrightarrow M - \gamma_E = -0.3157185 \tag{6}$$

($M = 0.2614972$ is the Mertens constant [2]); measured values $s = 1.1, 1.01, 1.001, 1.0001$ give -0.19374, -0.30252, -0.31439, -0.31559, tending to the limit (Figure 4). The limit of the slope is $\gamma_E + \sum_p \frac{\log p}{p(p-1)} = 0.577216 + 0.755366 = 1.332581$.

Reading the logarithm of this pole on the critical line is the Mertens $\log\log$ of [18], §7—"the slowest possible rate of divergence". The analytic guarantee that "the trembling of the primes reaches across the whole plane and has enough strength to touch the floor on the line" is engraved in this pole.

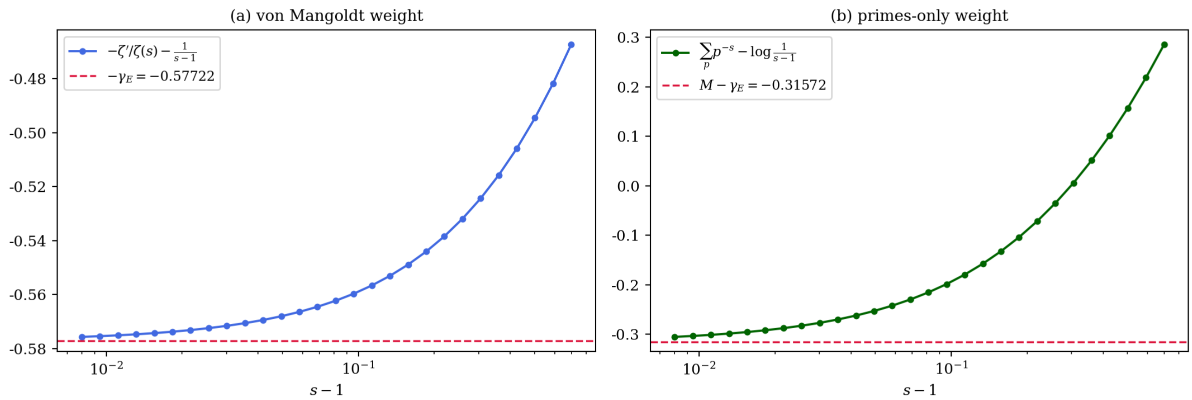

Figure 4: Pole = total intensity. (a) $-\zeta'/\zeta(s) - \frac{1}{s-1}$ converges linearly to $-\gamma_E$ as $s \to 1$. (b) The primes-only $\sum_p p^{-s} - \log \frac{1}{s-1}$ converges to $M - \gamma_E$.

6 The mirror and the pole are twins of a Poisson identity

We have now separately confirmed that the right wave is returned to the left (§4) and that the total intensity of the returned wave equals the pole (§5). This section shows that the two are **two terms of one and the same formula**.

Verifying the modular transform (the Poisson summation formula itself) of the theta series $\omega(x) = \sum_{n\geq 1} e^{-\pi n^2 x}$,

$$\omega\left(\frac{1}{x}\right) = \sqrt{x}\,\omega(x) + \frac{\sqrt{x}-1}{2}, \tag{7}$$

gives residuals 3.3×10^{-27} and 8.1×10^{-28} at $x=0.5, 0.25$. The two terms of this single formula play two roles. The factor $\sqrt{x}$ makes the mirror $s\leftrightarrow 1-s$—the "wave-returning device" verified in §4—and the constant term $\frac{\sqrt{x}-1}{2}$ (the compensation for the $n=0$ term of the Poisson sum) makes the pole. Indeed, Riemann's integral representation [1, 9]

$$\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\,\zeta(s)=\frac{1}{s(s-1)}+\int_1^{\infty}\big(x^{s/2-1}+x^{(1-s)/2-1}\big)\,\omega(x)\,dx \tag{8}$$

holds with residuals 2.1×10^{-28} and 2.0×10^{-27} at $s=0.6+7i$ and $s=-1.3+2.4i$. The second term on the right (the symmetric integral of the returned wave) is explicitly $s\leftrightarrow 1-s$ symmetric, and the pole pair $\frac{1}{s(s-1)}$ is also $s\leftrightarrow 1-s$ symmetric. The completion of zeta splits into "symmetric integral of the wave + symmetric pole pair", and both come out simultaneously from the same Poisson identity. **The pole and the mirror are not separate actors, but the twin outputs of one identity.**

This split is seen most vividly at the centre of the critical line. At $s=\frac{1}{2}$,

$$\pi^{-1/4}\Gamma\big(\tfrac14\big)\,\zeta\big(\tfrac12\big)=-3.97697=\underbrace{-4.00000}_{\text{pole pair }1/(s(s-1))}+\underbrace{+0.02303}_{\text{symmetric integral}},$$

i.e. the central value of the completed zeta is almost fully accounted for by the pole term, the wave integral carrying only 0.58% of the value (Figure 5).

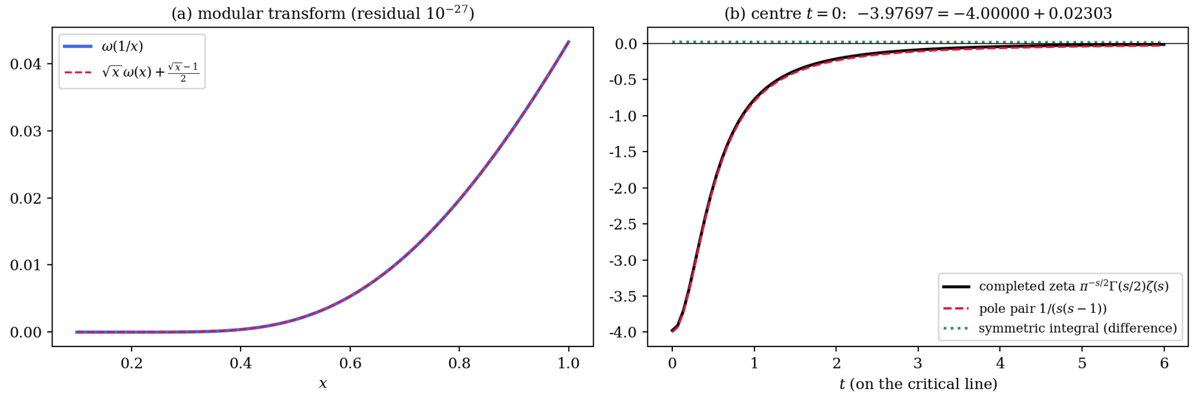

Figure 5: Twins of the Poisson identity. (a) Both sides of the modular transform overlap (residual 10^{-27})—the factor $\sqrt{x}$ makes the mirror, the constant term makes the pole. (b) Completed zeta on the critical line (black) = pole pair (red dashed) + symmetric integral (green dotted). At the centre the pole term carries almost all of the value.

7 Contrast with a genuine "no pole"—the lattice advances one site

Z_2 is a designed object. What then happens for a genuine "pole-free zeta"—a Dirichlet $L(s,\chi)$ of a primitive non-principal character? The answer is that **the right lattice does not stand; the lattice advances one site to the right**.

The completion of an even character ($\chi(-1)=1$) carries a $\Gamma(s/2)$ factor and has no pole pair $s(s-1)$. Because of this, a trivial zero appears where in ζ it was eaten by the mirror $s=0$ of the pole: $L(0,\chi)=0$. As a concrete example, take the real even character χ_5 modulo 5 (the quadratic-residue character); from the Hurwitz-zeta representation $L(s,\chi_5)=5^{-s}\big[\zeta(s,\frac15)-\zeta(s,\frac25)-\zeta(s,\frac35)+\zeta(s,\frac45)\big]$ one computes

$$L(0,\chi_5)=-6.5\times10^{-27}\approx 0,\qquad L(-2,\chi_5)=1.0\times10^{-26},\qquad L(-4,\chi_5)=-6.3\times10^{-26}$$

(exactly, from $\zeta(0,a) = \frac{1}{2} - a$, $\frac{3}{10} - \frac{1}{10} + \frac{1}{10} - \frac{3}{10} = 0$). Meanwhile $s = 1$ is an ordinary point,

$$L(1, \chi_5) = 0.43040894 = \frac{2}{\sqrt{5}} \log \frac{1+\sqrt{5}}{2} \quad \text{(8-digit agreement with the classical value)},$$

with $L(3, \chi_5) = 0.854825$, $L(5, \chi_5) = 0.965668$, and no zeros at $1, 3, 5, \ldots$ (Figure 6).

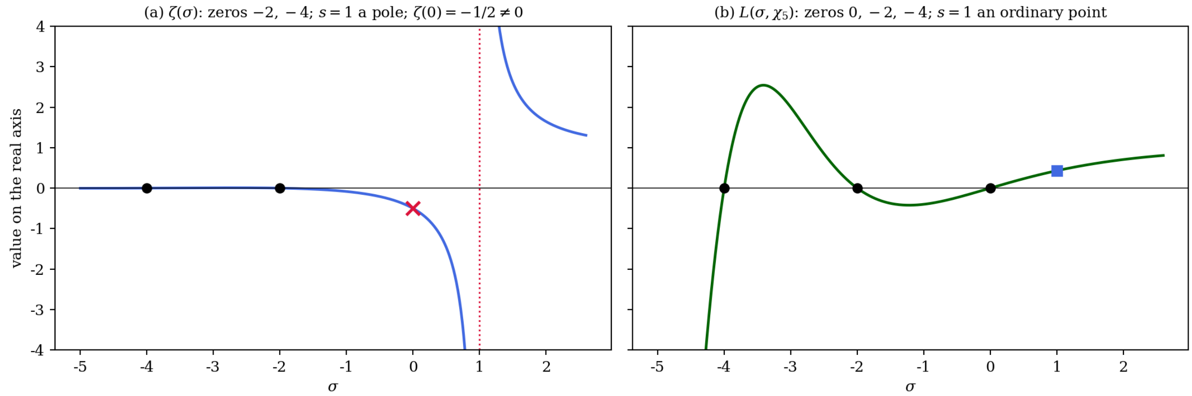

Figure 6: Comparison on the real axis. (a) $\zeta(\sigma)$: zeros at $-2, -4$, $s = 1$ a pole (dotted), $\zeta(0) = -\frac{1}{2} \neq 0$ ($\times$). (b) The pole-free $L(\sigma, \chi_5)$: zeros at $0, -2, -4$, the lattice advanced one site right, $s = 1$ an ordinary point (■). No zeros on the right.

The reason the right lattice does not stand is fundamental. **A function with an Euler product converging absolutely for $\sigma > 1$ can place no zero there—not even a trivial one.** Hence a right-side trivial zero can only be obtained in exchange for abandoning the Euler product in the right half-plane (the tie to the primes). Z_2 is exactly such a designed object (it does not belong to the Selberg class), and the one-sidedness of the Γ-factor of ζ is a demand from the side of the primes. In summary:

> **What removing the pole buys is only the single mirror site $s = 0$. The whole right lattice is the price of symmetrising the Γ-factor = cutting the tie to the primes in the right half-plane.**

And the pole of ζ stands there, unremoved, as a boundary marker of the divergence point $\sum_p p^{-1} = \infty$ of the Euler product.

8 The answer to Part I

The answer to Question I—if the pole $s = 1$ did not exist, would the right side return to the left?—in four sentences.

1. **The right lattice is already there.** The mirror lattice $\{1, 3, 5, \ldots\}$ is not empty but levelled to winding 0 by the exact cancellation of χ's poles against the mirror zeros. Only the first site has no cancelling partner and remains as the pole.
2. **Remove the pole and the right ponyo stands.** In Z_2 the pole is replaced by the first right trivial zero (winding $-1 \to +1$), and the right ponyo extends as the complete hue-inverted mirror of the left. Since Z_2 is real on the critical line, a left ponyo and its mirror meet the line between two consecutive zeros and serve those two zeros together.
3. **And the right wave is indeed returned to the left.** The trembling of the left ponyo agrees, with correlation 1.000000, with the prime field of the right half-plane returned by the functional equation through the mirror. All that remains is the archimedean t^{-3}.
4. **The returning device and the pole are twins.** $\sqrt{x}$ makes the mirror and the constant term of the same formula makes the pole pair. The total intensity of the returned wave is engraved in the pole as residue 1 and constant $-\gamma_E$.

In short, the pole is not "erasing the right-side ponyos". The pole is the twin, born of the same identity as the device that returns the right wave to the left, and it is the total intensity of the returned wave itself. And any attempt to remove the pole is paid for by the tie to the primes.

Part II

If the non-trivial zeros were not lined up on the critical line, would the twist of the complex plane vanish?

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9 Translating the question into a computation—the minimal counterexample model

In Part I we saw the source of the twist on the side of the pole. In Part II we move the receiver, that is, the zeros.

The question cannot be put to computation directly, because one cannot move the zeros of ζ by hand. So we translate it: **make a function that has a single pair of zeros moved off the line and is otherwise exactly the same as ζ. Preserve the functional equation, the real-axis symmetry, and the total zero count. Then measure where the phase portrait, the zero count, and the balance of the explicit formula differ from ζ.**

That is, we compute not "if ζ had a counterexample" but "how the phase portrait of a function with a counterexample looks". The former is a problem of proof, on which this note says nothing. The latter is a problem of pictures and numbers, and can be computed to the end.

9.1 Construction

Take the 6th and 7th zeros as $\gamma_6 = 37.5861781588\ldots$, $\gamma_7 = 40.9187190121\ldots$, and at the midpoint height $t_c = (\gamma_6 + \gamma_7)/2 = 39.2524485855\ldots$ place a mirror pair straddling the critical line,

$$\rho_L = \tfrac{1}{2} - \delta + i\,t_c, \qquad \rho_R = \tfrac{1}{2} + \delta + i\,t_c, \qquad \delta = 0.3.$$

Using the rational function

$$G(s) = \frac{(s-\rho_L)(s-\rho_R)(s-\bar\rho_L)(s-\bar\rho_R)}{(s-\rho_6)(s-\rho_7)(s-\bar\rho_6)(s-\bar\rho_7)}, \qquad \rho_n = \tfrac{1}{2} + i\gamma_n, \tag{9}$$

define

$$F(s) := \zeta(s)\,G(s). \tag{10}$$

The poles of G sit exactly on the zeros of ζ, so they cancel, and F is regular in the same range as ζ except for the simple pole at $s = 1$. The zeros of F are ρ_6, ρ_7 replaced by the mirror pair; everything else (the trivial zeros and all other non-trivial zeros) coincides with ζ. It is a designed object in the same sense as Z_2.

9.2 What is preserved

(i) Functional equation. Both the four zeros and the four poles have the form $\{\rho, \bar\rho, 1-\rho, 1-\bar\rho\}$, so $G(s) = G(1-s)$ holds identically and $F(s) = \chi(s)F(1-s)$ holds in exactly the same form as for ζ. Measured (working precision 25 digits, 4 points): relative residuals 1.5×10^{-25} to 5.2×10^{-25}.

(ii) Real-axis symmetry. The zeros and poles of G come in conjugate pairs, so $F(\bar{s}) = \overline{F(s)}$ and $\Xi_F(t) = \xi_F(\frac{1}{2}+it)$ is real ($\xi_F(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)F(s)$). Measured: $|\operatorname{Im} Z_F|/|Z_F| \leq 2.0 \times 10^{-25}$ on $t \in [36.5, 42.5]$, where $Z_F(t) = e^{i\theta(t)}F(\frac{1}{2}+it) = Z(t)G(\frac{1}{2}+it)$.
(iii) Winding. Measured on a circle of radius 0.12, F has $+1$ at both ρ_L, ρ_R and 0 at the original ρ_6, ρ_7. The new zeros are both simple, and the wedge rule of [18], §3.3—one wedge of opening 90° per colour quadrant, four in all—holds unchanged. The *neighbourhood* picture of a zero is indistinguishable from that of ζ.
(iv) Total zero count. As seen in §12.2, the total zero count measured by height is unchanged.

One thing, however, is not preserved: F has no Euler product, because G is a rational function with no connection to the primes. As we saw in Part I §7, this is not an accident but a structural circumstance, treated head-on in §14 and §18.

10 The local picture—inverted grounding

Figure 7 is the overall picture. Left and right, neither the *number* nor the *spacing* of the coloured bands has changed. What changed is where the ponyos land.

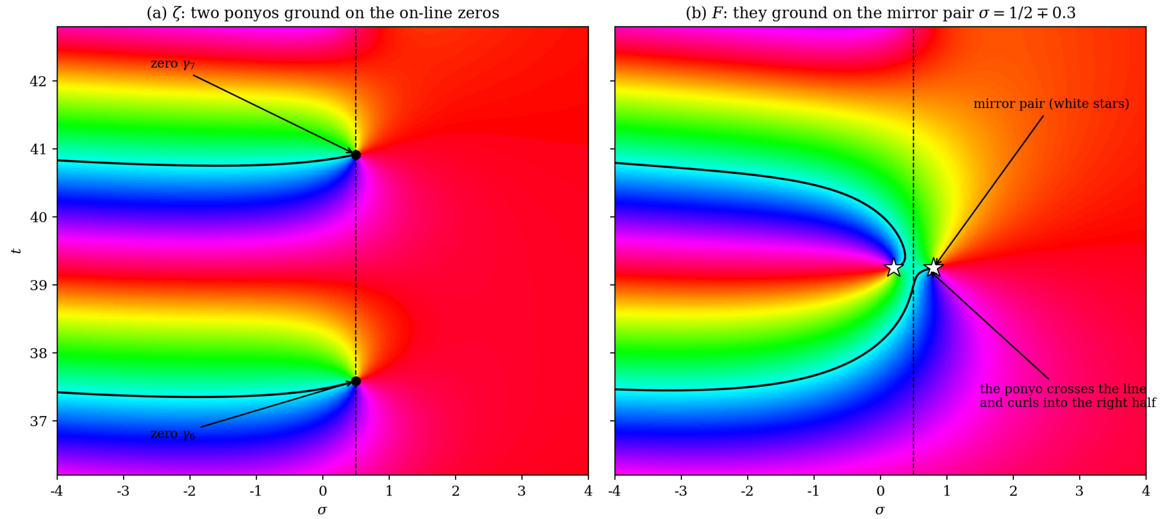

Figure 7: Comparison of phase portraits. Black curves are the cyan cores $\{\operatorname{Im} = 0, \operatorname{Re} < 0\}$. (a) ζ: two ponyos flow in from the left and ground straight onto the on-line zeros γ_6, γ_7. (b) F: the under ponyo crosses the critical line into the right half-plane and grounds on ρ_R; the upper ponyo overshoots ρ_L, turns back before reaching the line, and grounds on ρ_L from the right (white stars: the mirror pair).

In ζ (Figure 7a) the two cyan ponyos heading for γ_6 and γ_7 ground almost perpendicular to the critical line and end. There is no cyan core to the right of the line. Indeed, scanning $\sigma = 0.52, 0.6, 0.7, 0.8, 1.0, 1.5$ in the window $t \in [36.6, 42.4]$ finds not a single cyan core of ζ. (The curves $\operatorname{Im} \zeta = 0$ are the classical objects studied by Speiser [5] and, as "X-rays", by Arias de Reyna [14].)

In F (Figure 7b, zoom in Figure 8) the wiring of the same two ponyos changes: one crosses the line, the other overshoots its zero and turns back before the line. Table 2 gives the heights of the cyan cores at each σ.

The storyline one reads off (numbers as in Figure 8):

1. **The upper ponyo overshoots and curls back.** The upper ponyo flowing in from the left is still at $t = 40.03$ at $\sigma = 0$, then descends steeply, passes over the left zero ρ_L ($\sigma = 0.2$), and turns back at $\sigma^* = 0.37254$ (value pinned to 10^{-5} by bisection)—*still to the left of the critical line*: at $\sigma = 0.45$ only one core remains (Table 2), so this ponyo never reaches $\sigma = \frac{1}{2}$.

Table 2: Heights t of the cyan cores $\{\mathrm{Im} = 0,\ \mathrm{Re} < 0\}$ (window $t \in [36.6, 42.4]$). For ζ, two cores come up to the line and end. For F, they split into three past $\sigma = 0.2$, two merge and vanish at $\sigma^* = 0.37254$, and one remaining core reaches $\sigma = 0.8$. A dash (—) means the column was not scanned at that σ; "none" means scanned and no core found.

σ	cores of ζ	cores of F
-4.0	37.420, 40.834	37.464, 40.796
-2.5	37.358, 40.768	37.462, 40.678
-1.0	37.371, 40.762	37.666, 40.471
0.0	37.472, 40.835	38.163, 40.031
0.25	—	38.434, 39.267, 39.755
0.35	—	38.594, 39.334, 39.548
0.45	37.572, 40.909	38.827
0.52	none	39.066
0.70	none	39.227
0.775	none	39.241
0.80	none	none (grounds on ρ_R and ends)

2. **And it grounds on the left zero from behind.** The turned-back ponyo comes back leftward and lands on $\rho_L = 0.2 + i\,t_c$ *from the right*. Since in ζ a ponyo always enters a zero from the left, this means the local orientation of the phase portrait has been inverted.
3. **The under ponyo crosses the line.** The under ponyo crosses the critical line exactly at $t = 38.9992$.
4. **And it grounds on the right zero.** The crossing ponyo lands on $\rho_R = 0.8 + i\,t_c$. The penetration depth into the right half-plane is, of course, exactly $\delta = 0.3$, because the ponyo must reach the zero.

The neighbourhood of a zero itself is, as seen in §9.2 (iii), indistinguishable from ζ. The wedges are four, opening 90°, as before. What is broken is **the wiring between zeros**. In ζ the "one ponyo, one zero" correspondence can be read uniquely from both the ponyo and the zero side ([18], §5.1). In F the count still matches, but to trace which ponyo lands on which zero one must follow a path that either crosses the line (the under ponyo) or overshoots its zero and turns back (the upper ponyo).

In other words, what breaks first in the phase portrait when the Riemann hypothesis fails is not the picture of the zeros, but **the picture of the paths to the zeros**.

11 The grounding orientation—the ±90° rule is a degenerate case of the supplementary law

The sister paper [18], §5.2, showed two things about the grounding orientation. For ζ, $\arg \zeta'(\rho_n)$ drifts near 0°, a typical rule only; but for the completed zeta ξ, $\arg \xi'(\rho_n) = \pm 90.0000°$ holds as an *exact* alternation. This is a consequence of Ξ being real on the line (we reproduce $-90.0000°$ for ρ_6).

What becomes of this law when a zero leaves the line? The answer is that it does not vanish but is replaced by a broader law. Differentiating $\xi_F(s) = \xi_F(1-s)$ gives $\xi_F'(1-s) = -\xi_F'(s)$, and since $\rho_R = 1 - \overline{\rho_L}$,

$$\xi_F'(\rho_R) = -\overline{\xi_F'(\rho_L)} \quad \Longrightarrow \quad \begin{cases} \arg \xi_F'(\rho_L) + \arg \xi_F'(\rho_R) \equiv 180° \\ |\xi_F'(\rho_L)| = |\xi_F'(\rho_R)| \end{cases} \tag{11}$$

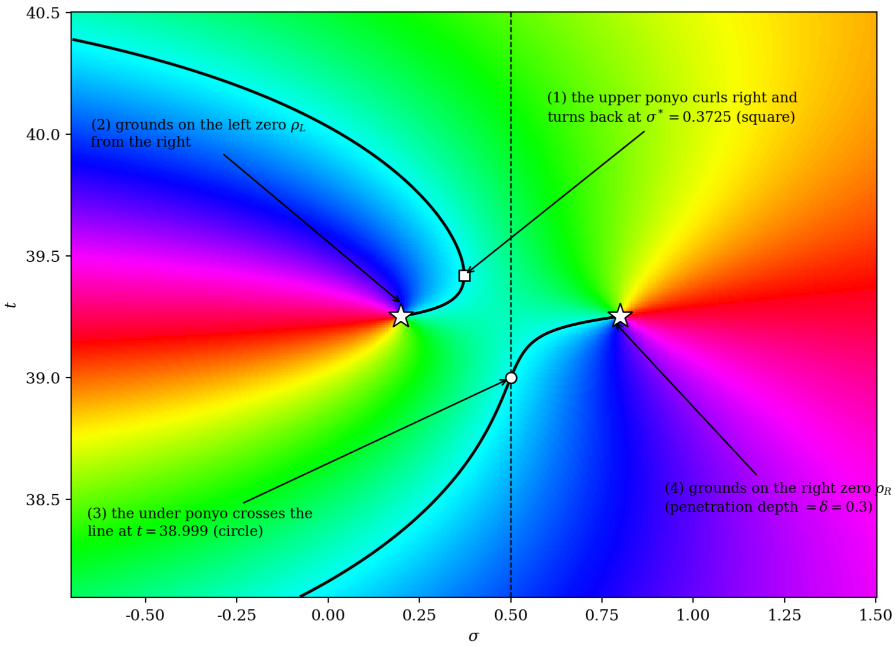

Figure 8: Zoom of the phase portrait of F. The upper ponyo overshoots ρ_L, turns back at $\sigma^* = 0.3725$ (square; to the left of the critical line), and grounds on the left zero ρ_L from the right. The under ponyo crosses the line at $t = 38.999$ (circle) and grounds on the right zero ρ_R. Penetration depth $\delta = 0.3$.

holds *exactly*. Measured:

$$166.1419627^\circ + 13.85803732^\circ = 180.0000000^\circ, \qquad \frac{|\xi_F'(\rho_L)|}{|\xi_F'(\rho_R)|} = 1.0000000.$$

The two groundings forming the mirror pair are exactly equal in strength and supplementary in orientation.

And in the limit $\delta \to 0$, $\rho_L = \rho_R = \rho$, and this becomes $2 \arg \xi'(\rho) \equiv 180^\circ$, i.e. $\arg \xi'(\rho) = \pm 90^\circ$. What appeared in the sister paper as the "exact alternation rule" was thus the **degenerate form of the supplementary law, holding only when the zero sits on the line**. In the language of the phase portrait, the grounding of ξ dutifully alternates green→blue and blue→green precisely because the mirror pair is collapsed to a single point.

For F itself, $\arg F'(\rho_L) = 165.641^\circ$ and $\arg F'(\rho_R) = -12.111^\circ$, with no simple law; as for ζ, the presence or absence of the winding factor $e^{-i\theta}$ separates the two.

12 The critical line loses its grounding—the completed Lehmer pair

12.1 The sign change of Z disappears

From $G(s) = G(1-s)$ and real-axis symmetry, G is *real* on the critical line (measured: $|\operatorname{Im} G|/|G| \le 8.1 \times 10^{-32}$). Hence $Z_F(t) = Z(t)\, G(\frac{1}{2} + it)$ is a product of two real functions. Here $G(\frac{1}{2} + it)$ has simple poles at $t = \gamma_6$ and $t = \gamma_7$ and is negative between them. The sign change of Z occurs exactly at those two points, so the poles and the sign changes cancel precisely, and Z_F does not change sign.

The numerics confirm this (Figure 9). $|Z_F|$ takes its minimum 0.0586426 at $t = 39.255431$, but does not touch zero there. It falls to 3.2% of the nearby typical value ($|Z| \sim 1.8$) and lifts back.

This is the very picture of a Lehmer pair [8, 12]. A Lehmer pair is the phenomenon of two zeros approaching abnormally so that the extremum of Z grazes the zero line; in [18], §8.2, we called it "a near-miss collision, the moment the symmetry creaks most". The present model is the case where that collision is **completed**. When two zeros actually collide and split into a mirror pair straddling the critical line, what remains on the line is only a shallow valley that does not reach zero.

Borrowing the phrase "strength enough to touch the floor on the line" from Part I §5, a counterexample is the line failing to touch the floor at just that one point.

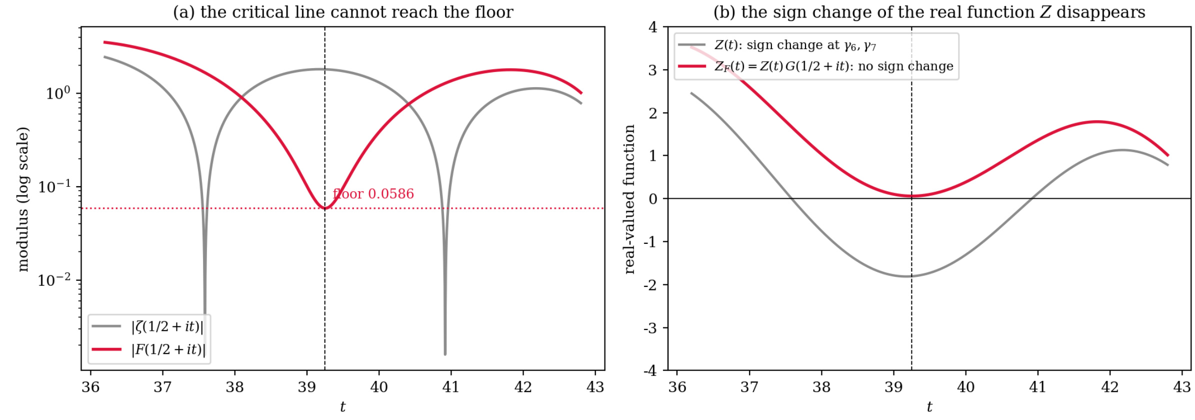

Figure 9: The completed Lehmer pair. (a) $|\zeta|$ touches zero at γ_6, γ_7, but $|F|$ stops at a floor of 0.0586. (b) Viewed as a real function, the poles of G exactly cancel the sign changes of Z, and Z_F does not change sign.

12.2 The winding is intact; the shift is borne by $S(T)$

What of the total zero count? The winding term $\theta(T)/\pi$ of $N(T) = \theta(T)/\pi + 1 + S(T)$ [11] is fixed by the Γ-factor alone and is the same for F and ζ. The difference appears in S. Following $S_F - S_\zeta = \frac{1}{\pi} \arg G$ continuously along the standard path from $s = 2$ up to $2 + iT$ and then to $\frac{1}{2} + iT$,

$$N_F(T) - N_\zeta(T) = \begin{cases} -1 & (\gamma_6 < T < t_c) \\ +1 & (t_c < T < \gamma_7) \\ 0 & (\text{otherwise}) \end{cases}$$

(measured: -1.0000 at $T = 38.0, 39.0$; $+1.0000$ at $T = 39.5, 40.0$; 0 at $T = 36.0, 42.0$).

That is, no zero has been lost or gained. Because the two zeros gathered at the same height t_c, the counting function lags by 1 for a while and then advances by 1. In [18], §6.2, we computed that the typical size of S is about $\sqrt{\log\log T}$ (Selberg [6]), a "sub-unit jitter" up to height 10^5. With one counterexample, a **unit box** of width $\gamma_7 - \gamma_6$ stands there. This is a size the prime fluctuation can make, but a shape it makes with difficulty.

13 The far field—the twist is a quadrupole and decays quickly

The heart of Question II was "does the twist vanish". Here, including the opposite possibility—that the twist might instead spread across the whole plane—we compute the far field.

For $|s| \to \infty$,

$$\log G(s) = -\frac{M_1}{s} - \frac{M_2}{2s^2} - \frac{M_3}{3s^3} - \cdots, \qquad M_k = \sum_{\text{zeros}} \rho^k - \sum_{\text{poles}} \rho^k. \tag{12}$$

Here M_1 is decisive. In a rearrangement preserving the functional equation, both zeros and poles move as the quadruple $\{\rho, \bar{\rho}, 1-\rho, 1-\bar{\rho}\}$. The sum of such a quadruple leaves only the real part and is always 2, so

$$M_1 = \sum \text{zeros} - \sum \text{poles} = 2 - 2 = 0 \tag{13}$$

holds **identically** (measured also exactly 0). That is, the mirror symmetry itself forbids the defect's dipole moment. It is no accident that the twist does not reach far.

The first surviving term is a quadrupole:

$$M_2 = 4\delta^2 + (\gamma_7 - \gamma_6)^2 = 0.360000 + 11.105829 = 11.465829, \qquad M_3 = 17.198743. \tag{14}$$

Interestingly, 96.9% of M_2 is the **vertical** term $(\gamma_7 - \gamma_6)^2$. Because a mirror pair must sit at the same height, moving zeros off the line necessarily entails a rearrangement of heights. The dominant component of the twist is not "shifting sideways" but "bringing together vertically".

Fixing σ and letting $t \to \infty$,

$$\arg G(\sigma + it) \simeq \frac{M_2\sigma - M_3/3}{t^3}, \tag{15}$$

predicting the coefficient -28.66457 at $\sigma = -2$. Measured, $\arg G \cdot t^3 = -29.081, -28.768, -28.690$ at $t = 800, 1600, 3200$, indeed converging to the prediction (Figure 10a).

Even if the far field is quiet, the near field moves. From Table 2, the positional shift of the ponyo is 0.30 at $\sigma = -1$, 0.10 at $\sigma = -2.5$, 0.04 at $\sigma = -4$. The ponyo spacing at this height is $2\pi/\log(t/2\pi) \approx 3.4$, so at $\sigma = -2.5$ the shift is 3% of the spacing and at $\sigma = -4$ only 1%. Compared with the size of the prime undulation measured in [18], §4.1 (11% of the spacing modulation at $\sigma = -2$), the effect of one counterexample buries itself in the trembling of the primes after only a few units to the left.

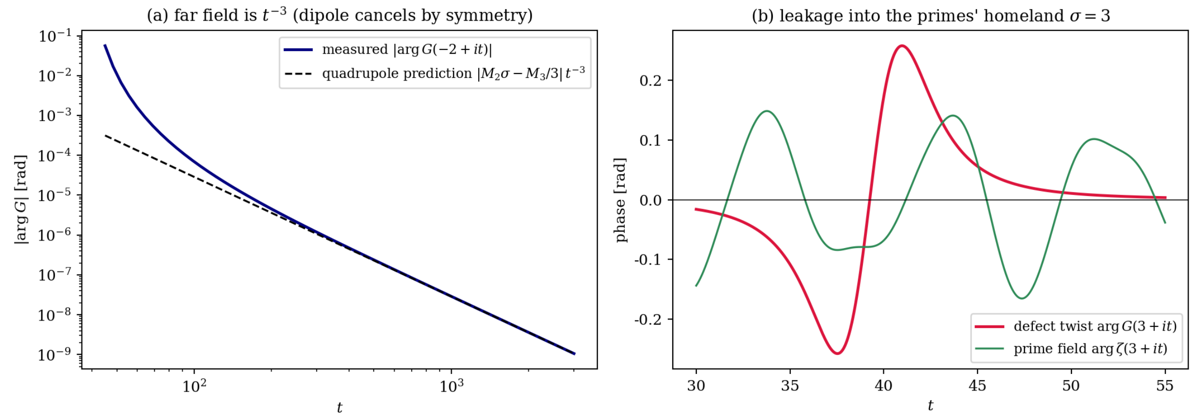

Figure 10: Reach of the twist. (a) The far field falls as t^{-3}, with the coefficient matching the quadrupole prediction $M_2\sigma - M_3/3$. (b) At the primes' homeland $\sigma = 3$, the defect twist (red, max 0.258 rad) exceeds the prime field itself (green, RMS 0.089 rad).

14 Leakage into the primes' homeland—and that it is a shadow of the model

For $\sigma > 1$, $\arg \zeta$ is fixed entirely by the Euler product. But in the model, since $G(s) = G(1-s)$, $|\arg G(3+it)| = |\arg G(-2+it)|$, reaching a maximum of 0.2581 rad ($t \approx 41$). The RMS of the prime field at the same place is 0.0885 rad, so the defect twist is 2.9 times the prime field

(Figure 10b). In the language of the phase portrait, onto the "uniform red sea" seen in Part I rides a bulge of a shape the primes cannot make.

But this must be read carefully. This leakage is a **property of the model, not of a ζ whose Riemann hypothesis has failed**. The real ζ has an Euler product wherever its zeros lie. So the phase field for $\sigma > 1$ remains the prime comb whether or not there is a counterexample. The leak is sealed from the outset.

Where, then, does the twist go? The Euler product refuses the distortion that would have spilled into the phase for $\sigma > 1$, and **transfers it to the error term of the prime staircase**. This is the subject of the next section. Part I §7 stated that any attempt to remove the pole is paid for by cutting the tie to the primes in the right half-plane. The present computation is its dual: any attempt to move zeros off the line cannot be paid on the Euler-product side and is paid on the arithmetic side.

15 The price paid in arithmetic—the budget of the explicit formula

This section alone is not about the coefficients of the model F, but the classical computation of "what becomes of the balance of the explicit formula if the zeros of ζ are assumed to have moved" (this switch of logic is noted again in §18).

In the explicit formula [1, 3]

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log 2\pi - \tfrac{1}{2}\log(1 - x^{-2}),$$

a zero $\rho = \beta + i\gamma$ supplies an oscillation of amplitude $2x^\beta/|\rho|$: $\sqrt{x}$ if $\beta = 1/2$, $x^{0.8}$ if $\beta = 0.8$.

x	mirror-pair term $2x^{0.8}/\lvert\rho_R\rvert$	on-line term $2\sqrt{x}/\lvert\rho_7\rvert$	ratio
10^2	2.03	0.489	4.15
10^4	80.7	4.89	16.5
10^6	3214	48.9	65.8

In [18], §8.3, we computed the envelope of the oscillation of all non-trivial zeros at $x = 10^4$ as $\sqrt{2x\sum|\rho|^{-2}} = 20.0$. With just *one* counterexample pair, that term becomes 80.7—four times the chorus of all zeros.

Figure 11 evaluates the zero sum with the first 100 zeros and normalises by $\sqrt{x}$. Whereas the zeros of ζ stay within 0.569 on $[10, 10^5]$, replacing one pair by the mirror pair reaches 1.841, growing neatly along the envelope $2x^{0.3}/|\rho_R|$. At $x = 10^{12}$ this envelope $2x^{0.3}/|\rho_R|$ exceeds 200.

This is nothing but the classical equivalence "Riemann hypothesis $\iff$ $\psi(x) = x + O(x^{1/2+\varepsilon})$" with a single zero singled out. What is, on the phase-portrait side, a small event of rewired ponyos, is on the arithmetic side a change in the order of magnitude of the error of the very distribution of the primes. The twist does not remain in the plane; it remains in the sequence of the numbers.

16 The pole does not notice—and its place in the phase diagram

In Part I §5 we confirmed that the total intensity of the prime waves has the closed form $-\gamma_E$. By how much does this constant move when one pair of zeros is moved? Since $-F'/F = -\zeta'/\zeta - G'/G$, the limit shifts to $-\gamma_E - (G'/G)(1)$. Measured:

$$G(1) = 1.0037292, \qquad (G'/G)(1) = -7.262 \times 10^{-6}, \qquad -\gamma_E - (G'/G)(1) = -0.5772084.$$

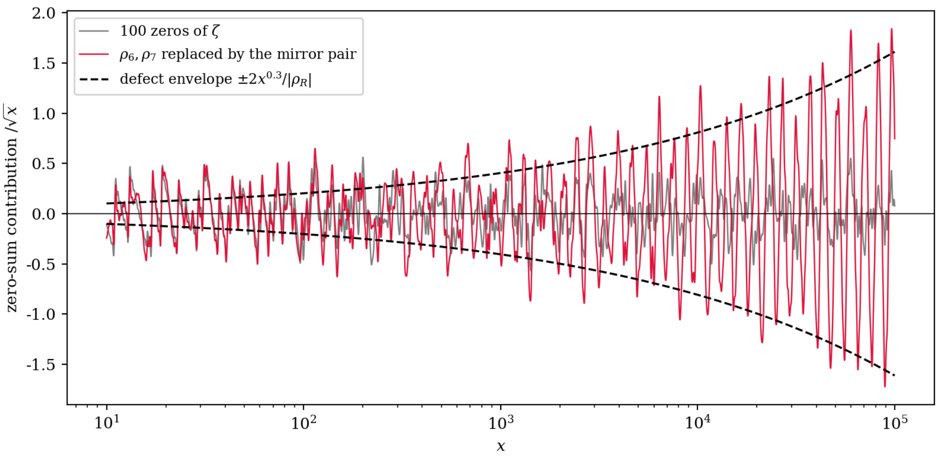

Figure 11: Budget of the explicit formula. The zero sum evaluated with 100 zeros, divided by $\sqrt{x}$. The zeros of ζ (grey) stay bounded, but replacing one pair by the mirror pair (red) grows along the defect envelope $\pm 2x^{0.3}/|\rho_R|$.

The shift is 7.3×10^{-6}, 0.0013% of γ_E. The measured sequence as $s \to 1^+$ (-0.567958, -0.575338, -0.576833 at $s = 1.05$, 1.01, 1.002) agrees to the fourth decimal with the table of Part I §5. The residue remains exactly 1, and the leading term $\psi(x) \sim x$ of the prime number theorem does not move either.

The reason is simple: the defect is at height 39, far from the pole $s = 1$. The decay law of the quadrupole in §13 is exactly its appearance from the pole's side. The "pole-quake" identity studied in Part I is robust against a local break on the line. **A counterexample does not shake the pole.**

Finally we check the relation to the "two guards" of [18], §7. At $\sigma = 0.8$,

$$\text{prime-noise variance } \tfrac{1}{2}\sum_{p\le 10^6} p^{-1.6} = 0.3674 \text{ (finite)}, \qquad \text{drift } (\tfrac{1}{2} - 0.8)\log\frac{t_c}{2\pi} = -0.5496.$$

The variance is finite, and the Bohr–Jessen limit distribution [4] has a density, so the probability that $\log|\zeta|$ reaches $-\infty$ (= zero) is 0. The model F is forcing this "probability-0 event" by hand. Hence the present computation can at most explain that a counterexample is unlikely, and never proves that it is impossible.

We also confirm the location in the language of the de Bruijn–Newman heat flow [7, 10]. In the mean-field pitchfork the order parameter follows $b = \sqrt{2(\lambda_c - \lambda)}$ ([18], §8.2). Setting $b = \delta = 0.3$ gives $\lambda_c - \lambda = 0.045$, so the system has stepped 0.045 inside the transition point, into the broken-symmetry phase. Combining the theorems RH $\iff$ $\Lambda \le 0$ and $\Lambda \ge 0$ (Rodgers–Tao [16]), the Riemann hypothesis holds iff $\Lambda = 0$. A counterexample means that ζ sits not exactly at the transition point but inside the broken phase. But this is only a local mean-field correspondence; Λ itself is not computed.

17 The answer to Part II

The answer to Question II—if the zeros were not lined up on the critical line, would the twist vanish?—again in four sentences.

1. **The twist does not vanish.** The winding is supplied by the archimedean factor, not by the zeros. Neither the number nor the spacing of the ponyos changes at all from ζ.
2. **Nor does the twist spread.** The functional equation identically forbids the defect's dipole moment, so the far field falls as a quadrupole t^{-3}. A few units to the left and the

twist buries itself in the prime undulation. The pole constant $-\gamma_E$ moves only at the 10^{-5} digit.

3. **What breaks is the wiring.** The under ponyo crosses the critical line and lands on the right zero inside the right half-plane; the upper ponyo overshoots the left zero, turns back at $\sigma^* = 0.3725$ before reaching the line, and grounds on that zero from behind. The line itself loses its grounding, and $|Z_F|$ stops at a shallow floor that does not reach zero. The $\pm 90^\circ$ rule that governed the grounding orientation is replaced by the supplementary law.
4. **The price is paid in arithmetic.** The $\sqrt{x}$-normalised error of the prime staircase begins to grow at $x^{0.3}$.

In short, the zeros are not the source of the twist but its receivers. Moving the receivers does not make the twist vanish. The balance of its not vanishing is settled on the side of the sequence of the primes.

Part III

Putting the two answers together

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18 One picture

The two questions, while carrying separate answers, fit into one and the same picture.

Who makes the twist? The pole and the archimedean factor. The constant term of the Poisson identity makes the pole, and the Γ-factor supplies the winding. As seen in Part I §5, the residue 1 and constant $-\gamma_E$ of the pole are the total intensity of the returned prime waves themselves.

Who returns the twist? The mirror. The factor $\sqrt{x}$ of the same Poisson identity makes $s \leftrightarrow 1-s$ and returns the prime field of the right half-plane to the left. The correlation 1.000000 of Part I §4 is the direct measurement of that return.

Who receives the twist? The zeros. Each ponyo grounds, one per zero, on a zero and ends. So moving the zeros does not make the twist vanish; only the manner of grounding changes—this was the answer of Part II.

Note that the two parts arrive at conclusions of the same shape. Under either "if", the Euler product sends the bill in the end.

> **The price of removing the pole.** What removing the pole buys is only the single mirror site $s = 0$. To stand the whole right lattice requires symmetrising the Γ-factor, which means cutting the tie to the primes in the right half-plane (Part I §7).
>
> **The price of moving the zeros.** On the phase side, the twist is merely rewired locally. But the Euler product passes no distortion into the phase for $\sigma > 1$, so the distortion is transferred to the error term $x^{0.8}$ of the prime staircase (Part II §15).

The Z_2 of Part I showed that "cut the tie to the primes and a lattice and ponyos stand on the right too". The F of Part II showed that "keep the tie to the primes and move the zeros, and the phase portrait is almost intact while the distribution of the primes breaks instead". Setting the two side by side, one sees the shape of ζ being squeezed doubly. The left–right asymmetry of the phase portrait (that there is no right lattice) is demanded by the Euler product; that the zeros line up on the line—if they truly do—is demanded by the $\sqrt{x}$ accuracy of the prime staircase. The pole and the zeros are the two endpoints of one and the same constraint, the Euler product.

Of course, this is not an explanation of "why the zeros stand on that line". The honest boundary line drawn by the sister paper [18] does not move here either. All that this note adds is a map of the terrain around that question, surveyed from the direction of two "ifs".

19 What is a theorem and what is not

Limitations. (a) This note contains no new mathematical result. Part I is entirely an organisation and numerical check of classical facts; Part II is elementary computation for an explicit model. (b) Both Z_2 and F are designed objects for explanation, and neither has an Euler product (neither belongs to the Selberg class). Hence F is not a picture of "ζ with the Riemann hypothesis false". The real ζ has an Euler product regardless of where its zeros lie, so the "leakage into the primes' homeland" of §14 is a shadow of the model. (c) Section 15 alone follows a different logic: it is the classical computation of the explicit-formula budget under the assumption that the zeros of ζ have moved. It is not about the coefficients of the model F. (d) The choice of the pair (ρ_6, ρ_7) and the offset $\delta = 0.3$ is arbitrary. The local picture (inverted grounding, supplementary law, quadrupole) is general, but individual numbers such as $\sigma^* = 0.3725$ and the floor 0.0586 are specific to the model. (e) Numbers such as windings, cancellations, and mirror residuals are finite-precision checks (8–30 digits), and the proofs lie on the side of the respective classical facts. The core tracking in the phase portrait is based on a finite grid (steps of order 10^{-3} in both σ and t). (f) The de Bruijn–Newman correspondence is a local mean-field analogy, not a computation of Λ. (g) Bibliographic details are given from memory and have not been checked against the originals, nor was a systematic literature search made. The level-curve analysis is likely to overlap with the X-ray analysis of Arias de Reyna [14]. References [18, 17, 19] are non-archival. (h) The computations are by short scripts reimplementable from the text, but the code is not released.

20 Concluding remarks

Setting the two "ifs" side by side and computing them, one sees clearly, split apart, who is doing what in the plane of ζ.

If the pole $s = 1$ did not exist—a lattice would stand on the right too, and the ponyos would extend as the complete mirror of the left with the hues swapped. But in the real ζ that lattice is levelled by cancellation, winding and all, and only the first site remains as the pole. And the right-side wave is not lost. The functional equation returns it to the left, and the returning device and the pole are twins born simultaneously of one Poisson identity. The total intensity of the returned trembling is engraved in the pole as residue 1 and constant $-\gamma_E$.

If the zeros were not lined up on the line—the twist of the plane neither vanishes nor spreads. The functional equation forbids the dipole, so the effect of the defect falls as a quadrupole t^{-3}, and the pole hardly notices. What breaks is the wiring just beside the critical line. One ponyo crosses the line into the right half-plane; the other overshoots its zero, turns back, and grounds from behind. The line loses its grounding, and a would-be Lehmer pair is completed there. And the price is paid outside the phase portrait, as the error term of the prime staircase.

The pole makes the twist, the mirror returns it, the zeros receive it. The two questions were two probes, from either end, of this single flow. Whichever end one moves, the one who sends the bill in the end is the Euler product.

Table 3: Status of the claims of this note.

claim	status
Part I	
neutralisation of the mirror lattice (χ's poles vs. mirror zeros; only $s = 1$ survives as pole)	classical (elementary consequence of the functional equation); winding checked numerically
properties of Z_2 (entire, mirror-symmetric, two-sided lattice, $Z_2'(1) = -1/(4\sqrt{\pi})$, sign alternation; reality on the critical line)	elementary; verified to $\sim 10^{-19}$
growth of the right ponyo (hue-inverted mirror, right comb equation, mirror amplitude match)	elementary (reflection formula); illustrated by figures
verification of the wave return (correlation 1.000000, residual $\propto t^{-3}$)	restatement of [18] eq. (7); verified numerically
limit $-\gamma_E$ and slope $\gamma_E^2 + 2\gamma_1$	classical (Laurent expansion); verified numerically
Poisson twins (modular transform and integral representation; pole pair also mirror-symmetric)	classical (Riemann [1]); verified to 10^{-27}
$L(0,\chi) = 0$ (even primitive character) = lattice advancing one site	classical [13]; verified numerically
Euler product $\Rightarrow$ no zeros for $\sigma > 1 \Rightarrow$ incompatible with a right lattice	classical (absolute convergence)
Part II	
F preserves functional equation, real-axis symmetry, simplicity of zeros	elementary (by construction); verified to 10^{-25}
supplementary law $\arg \xi_F'(\rho_L) + \arg \xi_F'(\rho_R) = 180°$, equality of $\lvert\xi_F'\rvert$	exact (consequence of $\xi_F(s) = \xi_F(1-s)$); verified numerically
$\pm 90°$ alternation is the $\delta \to 0$ degeneration of the supplementary law	exact; reinterpretation of [18] §5.2
Z_F loses its sign change (floor 0.0586)	exact (reality of G on the line and pole positions); verified numerically
the unit box of $N_F - N_\zeta$	exact (argument principle); verified numerically
$M_1 = 0$ (vanishing dipole)	exact (sum of the mirror quadruple = 2)
far field $\arg G \simeq (M_2\sigma - M_3/3)t^{-3}$	elementary (expansion); verified for $t \leq 3200$
inverted grounding; turning point $\sigma^* = 0.37254$ (left of the line); crossing $t = 38.999$	numerical (finite grid, bisection); illustrated by figures
leakage into $\sigma > 1$ (0.258 rad)	property of the model; does not transfer to real ζ (§14)
error term of $\psi(x)$ becoming $x^{0.8}$	classical (explicit formula); visualised numerically
shift of the pole constant 7.3×10^{-6}	elementary; verified numerically
$\lambda_c - \lambda = \delta^2/2$	mean-field analogy; Λ not computed
the narrative of “twist”, “wave return”, “anti-grounding”, “inverted grounding”	metaphor; mathematical content confined to the formulae above
the truth of the Riemann hypothesis	this note says nothing

21 Changes in the corrected version (2 September 2026)

The following errors and internal contradictions of the 19 August version were corrected; no numerical value was changed.

1. **Mirror lattice (Abstract (1), §1 Question I, §2).** The old text called $\{0, -2, -4, \dots\}$ "the trivial-zero lattice" and, inconsistently, said that reflecting $\{-2, -4, -6, \dots\}$ gives $\{1, 3, 5, \dots\}$. The reflection of $\{-2, -4, -6, \dots\}$ under $s \mapsto 1 - s$ is $\{3, 5, 7, \dots\}$; the site $s = 1$ is the mirror of $s = 0$, where $\zeta(0) = -\frac{1}{2} \neq 0$. The lattice $\{0, -2, -4, \dots\}$ is now described as the pole lattice of $\Gamma(s/2)$, of which only $-2, -4, \dots$ are trivial zeros.
2. **Turning point of the upper ponyo (Abstract (8), §10, Figures 7 and 8, §17, §20).** The old text said the upper ponyo "crosses the critical line" and turns back at $\sigma^* = 0.37254$. Since $\sigma^* < \frac{1}{2}$ and Table 2 shows a single core at $\sigma = 0.45$, the upper ponyo never reaches the critical line: it overshoots ρ_L ($\sigma = 0.2$), turns back at σ^*, and grounds on ρ_L from the right. Only the under ponyo crosses the line (at $t = 38.999$) and enters the right half-plane, where it grounds on ρ_R. All passages have been reworded accordingly.
3. **Grounding of Z_2 (§3.2, Figure 2, §8).** Z_2 is real on the critical line, so a left cyan core and its mirror cannot both enter a simple zero from the sides; they meet at a saddle point on the line between two consecutive zeros, and one mirror pair serves two zeros (consistently with $\Delta \arg Z_2(-2 + it) \approx -\theta(t)$, half the winding of ζ). The caption of Figure 2, which spoke of a pair "grounding on a shared zero", was corrected, an explanatory paragraph was added, and Figure 2 was redrawn with the cyan cores traced (the saddle points lie at $t = 29.31$, 36.58, 42.62).
4. **Abstract (9), §17.** "$|Z|$" was replaced by "$|Z_F|$": the floor 0.0586 is a property of the model, not of ζ.
5. **§4.** The origin of the correction term $3.125/t$ (the $O(1/t)$ Stirling term of $\arg \chi(\sigma + it)$ at $\sigma = -2$) is now stated.
6. **Table 2.** The meaning of "—" (not scanned) versus "none" (scanned, no core) is now explained.
7. **§15.** "At $x = 10^{12}$ this ratio exceeds 200" was ambiguous; it now names the quantity (the envelope $2x^{0.3}/|\rho_R|$).
8. **References.** The entries [2, 5, 9, 11, 15] were listed but never cited; citations were added where they belong.

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